「システム制御理論特論」(前半)

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Introduction

This lecture explains the nonlinear control theory, especially Lyapunov methods.

Fundamentals of nonlinear system   Expressions of nonlinear system, vector field on manifold, existence and uniqueness of ordinary differential equation

Exact linearization   Exact I/O linearization，zero dynamics，semi-global stabilization and peaking，Exact state linearization

Lyapunov method   Lyapunov function, dissipative inequality, passivity, Sontag-type controller, input-to-state stability (ISS)

(参考) 小林先生担当分: ハイブリッドシステムの基礎 (ハイブリッドシステムのモデル), **混合論理動的システムモデル** (命題論理の線形不等式表現), **モデル予測制御** (有限時間最適制御問題，混合整数二次計画問題，オンライン解法，オフライン解法)

This document

You can download this documentation (PDF) at

http://stlab.ssi.ist.hokudai.ac.jp/yuhyama/lecture/tokuron/

This document will be revised frequently.

Nonlinear ordinary differential equation with input

Continuous-time nonlinear system:

- Input-Affine System (入力アフィン系)

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$
$$y = h(x)$$

x ··· state, $u(\in \mathbb{R}^m)$ ··· input, $y(\in \mathbb{R}^\ell)$ ··· output

- General Nonlinear System

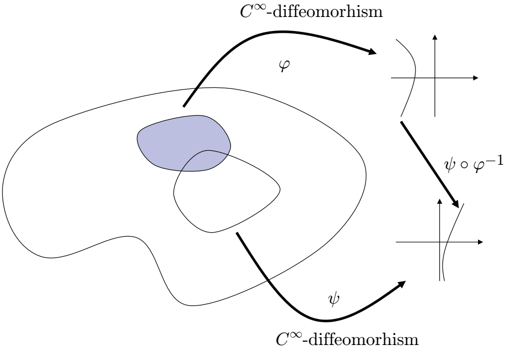
$$\dot{x} = f(x,u)$$
$$y = h(x)$$

x ··· state, $u(\in \mathbb{R}^m)$ ··· input, $y(\in \mathbb{R}^\ell)$ ··· output

State-space of nonlinear systems

System state x denotes a point of an n-dimensional C^∞ manifold (C^∞ 多様体).

- C^∞ manifold: Hyper surface having a uniform dimension.
- Infinite times differentiations are available on the manifold

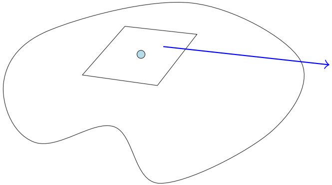


- For each point of the manifold M, there exists a local neighborhood that is C^∞ diffeomorphic to $\mathbb{R}^n$(= n-dimensional Euclidean space). ⇒ Local coordinate

- There exists a C^∞ diffeomorphism between two neighborhoods, where its domain is the intersection of the two set. ⇒ Coordinate transformation (座標変換)
- M can be covered by some neighborhoods.

Tangent space

[Question] What space does $\dot{x}$ belong to?



Tangent space (接平面) of a point p is diffeomorphic to $\mathbb{R}^n$.

$$T_pM \approx \mathbb{R}^n$$

Let TM denote the union of the tangent spaces of all points.

In the local sense, both of x and $\dot{x}$ are n-dimensional. In the global sense, we recognize that

$$x \in M, \quad (x, \dot{x}) \in TM.$$

- Practically, it is useful to introduce a local coordinate to x which derives a natural coordinate on the space of $\dot{x}$.
- In a local sense, TM_U has a structure of direct product $M_U \times \mathbb{R}^n$. However, it is not correct globally. $\Rightarrow$ TM may be 'twisted.'

Tangent space (Example of 3D-rotation)

An attitude of a rigid body can be expressed by an orthogonal matrix with a positive determinant $R \in SO(3)$.

$$R^\top R = I, \qquad \det R = 1$$

$\Rightarrow$ The degree of freedom is three

Kinematic equation:

$$\dot{R} = S(\omega)R$$

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

$\dot{R}$ belongs to a 3-D space when R is fixed. It is parameterized by a vector space $\omega = (\omega_1, \omega_2, \omega_3)^\top$.

$\dot{R}$ cannot be determined by only ω. The value of R is required to determine it.

Conclusion of "nonlinear system expression"

- The input-affine systems are well studied as expressions of nonlinear dynamical systems.
- The state x is a point of a n-dimensional differential manifold, and it is not a vector space generally.
- However, $\dot{x}$ belongs to a n-dimensional Euclidean space when x is fixed.
- The right-hand side of $\dot{x} = f(x)$ is called a vector field (ベクトル場).

Solution of ODE

Problem

Given an ODE

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

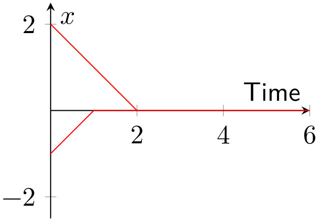
with an initial condition $x(0) = x_0$, find a solution $x(t)$ $(t \geq 0)$ of the ODE.

- Does the solution exist? (解の存在性)
- Suppose that a solution exists. Is it a unique solution? (解の唯一性)

ODE having no solution

An example of ODE having no solution:

$$\dot{x} = \begin{cases} -1 & (x \geq 0) \\ 1 & (x < 0) \end{cases}$$

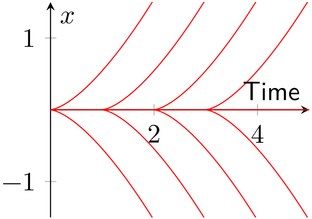


At a glance, the state tends to $x = 0$ in a finite time. After x gets to 0, the derivative $\dot{x}$ should be also zero.
However, $(x, \dot{x}) = (0, 0)$ does not satisfy the original ODE.

ODE having multiple solutions

An example of ODE having multiple solutions:

$$\dot{x} = \mathsf{sgn}(x)\sqrt[3]{|x|}$$



The ODE with an initial condition $x(0) = 0$ has an infinite number of solutions.

Lipschitz condition

Definition

A map $f(x)$ satisfies the Lipschitz condition on a connected open set U, if there exists $M(>0)$ such that

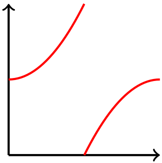
$$\|f(x_1) - f(x_2)\| \leq M\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in U.$$

$\Rightarrow$ A weak concept of differentiability

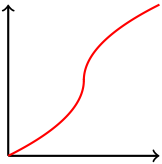
If $f(x)$ is Lipschitz on the universal set (e.g. $\mathbb{R}^n$), $f(x)$ is called **globally Lipschitz**.

If for each x, there exists a neighborhood U_x such that $f(x)$ is Lipschitz on U_x, $f(x)$ is called **locally Lipschtz**.
Note that the values of M may be different for the neighborhoods.

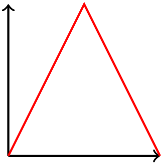
Lipschitz condition (example)



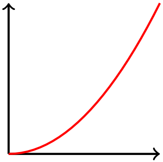
No Lipschitz (discontinuous)



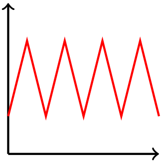
Continuous but no Lipschitz



Lipschitz



Locally Lipschitz but not globally Lipschitz ($y = x^2$)

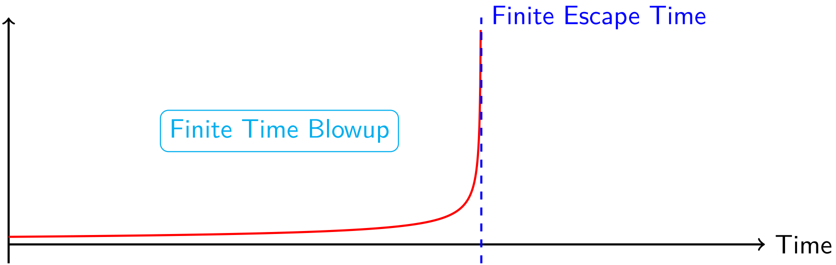


Globally Lipschitz

Sufficient conditions of the existence of a unique solution

- (Theorem of Picard-Lindelöf) If $f(x)$ is locally Lipschitz, there exists a positive T such that the ODE $\dot{x} = f(x)$ with an initial condition $x(0) = x_0$ has a unique solution for $0 \leq t \leq T$. The value of T depends on the initial value x_0.
- (Extension of the solution) If $f(x)$ is globally Lipschitz, $\dot{x} = f(x)$, $x(0) = x_0$ has a unique solution globally, i.e., for $-\infty < t < \infty$.

■ An example of ODE having local solution: $\dot{x} = x^3$ (Locally Lipschitz)



In this case, the solution diverges in finite time.

Sufficient conditions of the existence of solutions

- (Peano existence theorem) If the uniqueness of the solution is not required, we can weaken the condition of the Picard-Lindelöf's theorem, i.e., only the continuity of $f(x)$ is necessary.
- A variance of the Peano existence theorem for time-variant ODEs exists.
- Carathéodory's existence theorem gives a further generalization.
- For more details, see the following famous book of Coddington & Levinson:

  E.A. Coddington, N. Levinson: *Theory of Ordinary Differential Equations*, McGraw-Hill (1955).

Exact linearization

Nonlinear plant $+$ Nonlinear control law $\rightarrow$ Linear closed-loop system

- Finally obtained linear system has a different coordinate of state from the original nonlinear system. (Nonlinear coordinate transformation)
- This method is based on no approximation, and therefore it is called **exact** linearization.
- The exact-linearization technique includes 'exact input-output linearization' (厳密入出力線形化) and 'exact state-space linearization' (厳密状態空間線形化).
- As a mathematical tool, we use Lie derivative. Moreover, we also utilize Lie bracket and Frobenius theorem for the state-space linearization cases.

The case of mechanical system

Mechanical system (e.g. robots)

$$M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta) = u$$

We can apply a feedback

$$u = c(\theta, \dot{\theta}) + g(\theta) + M(\theta)v$$

to this system. The closed-loop system is linearized as $\ddot{\theta} = v$.

- This is well-known technique in Robotics.
- This method cancels nonlinear term via feedback.

Can we apply this method to general cases?

Concept of exact I/O linearization

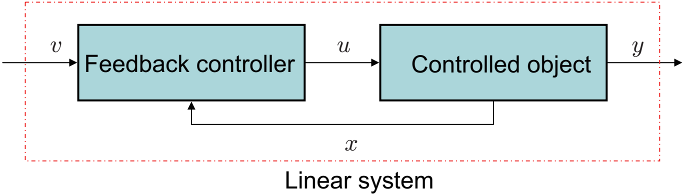
For the system

$$\dot{x} = f(x) + G(x)u$$
$$y = h(x)$$

we use a state feedback

$$u = \alpha(x) + \beta(x)v$$

to exactly linearize the I/O behavior from v to y.



Lie derivative

Main mathematical tool of exact linearization = Lie derivative (リー微分)
Lie derivative operators (リー微分作用素) can be applied to general tensors, but in our case we use only a subset.

- **Lie derivative that is applied to usual functions**
  (Local coordinate expression)

  $h(x)$: $M \to \mathbb{R}$ (a function of x)
  $f(x)$: $M \to TM$ (vector field)

$$(L_f h)(x) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} f_i(x) = \frac{\partial h}{\partial x}(x) f(x)$$

- **Repeat of the operators**

$$(L_g L_f h)(x) = (L_g(L_f h))(x)$$
$$(L_f^k h)(x) = (\underbrace{L_f(L_f(\cdots(L_f}_{k-\text{times}} h)\cdots)))(x)$$

Practical meaning of Lie derivative

Suppose that $x(t)$ moves along a solution of the system without input

$$\dot{x} = f(x)$$

Consider a function $y = h(x)$. Its time derivative can be calculated as

$$\frac{dy}{dt} = \frac{\partial h(x)}{\partial x}\frac{dx}{dt} = \frac{\partial h(x)}{\partial x}f(x) = (L_f h)(x)$$

$L_f h$ is the time derivative of $h(x)$, which is a function of x, along the trajectory of $\dot{x} = f(x)$.

Differentiation of output by t

Consider a single-input single output system:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

Differentiation of output by t

$$\dot{y} = \frac{\partial h}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial h}{\partial x}(f(x) + g(x)u)$$
$$= (L_{f+gu}h)(x, u) = L_f h(x) + L_g h(x)u$$

Applying L_{f+gu} to $h(x)$, which is a function of x, is equivalent to obtaining the time derivative of $h(x)$

Linear cases:

$$\dot{y} = C(Ax + Bu) = CAx + CBu$$

Is twice differentiation possible?

Question

Does

$$\frac{d^k y}{dt^k} = L_{f+gu}^k h$$

hold generally?

The answer is NO. It is because the result of the first derivative

$$\dot{y} = (L_{f+gu}h)(x(t), u(t))$$

is not a function of solely x. It becomes **a function of x and u** generally.

$$\ddot{y} = \frac{d}{dt}(L_{f+gu}h)(x, u) = L_{f+gu}L_f h + L_{f+gu}L_g h \cdot u + \dot{u} \cdot L_g h$$

$\rightarrow$ If $L_g h$ is nonzero and $u(t)$ is nondifferentiable, $y(t)$ is not twice differentiable.

To differentiate $y(t)$ twice, $L_g h$ should be zero generally.

If $L_g h \neq 0$

- If in the time-derivative of the output

$$\dot{y} = L_f h(x) + L_g h(x) \cdot u$$

the coefficient of u is nonzero, i.e., $(L_g h)(x) \neq 0$, then

$$u = \frac{-L_f h(x) + v}{L_g h(x)} \quad \Rightarrow \quad \dot{y} = v$$

I/O behavior from the new input v to y is linearized = Canceling nonlinear terms

- For practical cases, a further feedback with pole assignment is required.

However, there exist cases with $L_g h = 0$.
For example, the derivation of a physical position gives a velocity, which is a state and includes no input term.

$\Rightarrow$ Twice differentiation

Twice differentiation of y

If $L_g h = 0$, y can be differentiated twice.

Assumption: $L_g h = 0$

$$\ddot{y} = L_{f+gu} L_f h = L_f^2 h(x) + L_g L_f h(x) \cdot u$$

$$\Downarrow$$

If $L_g L_f h(x) \neq 0$, the system can be linearized by

$$u = \frac{-L_f^2 h(x) + v}{L_g L_f h(x)} \quad \Rightarrow \quad \ddot{y} = v$$

Three-times differentiation...

- **Assumption:** $L_g h = 0$, $L_g L_f h = 0$

$$\frac{d^3 y}{dt^3} = L_{f+gu} L_f^2 h = L_f^3 h(x) + L_g L_f^2 h(x) \cdot u$$

$$\Downarrow$$

If $L_g L_f^2 h(x) \neq 0$, the system can be linearized by

$$u = \frac{-L_f^3 h(x) + v}{L_g L_f^2 h(x)} \quad \Rightarrow \quad \frac{d^3 y}{dt^3} = v$$

- ...and so forth on.

Relative degree

Definition of relative degree (相対次数)

The output y has a relative degree ρ at a point x_0, if there exists a neighborhood U_{x_0} of x_0 such that

$$(L_g L_f^i h)(x) = 0, \quad i = 0, \ldots, \rho - 2, {}^\forall x \in U_{x_0}$$
$$(L_g L_f^{\rho-1} h)(x_0) \neq 0$$

If a relative degree ρ exists, the output can be differentiate ρ-times.

$$\dot{y} = L_f h(x)$$
$$\ddot{y} = L_f^2 h(x)$$
$$\vdots$$
$$\frac{d^{\rho-1} y}{dt^{\rho-1}} = L_f^{\rho-1} h(x)$$
$$\frac{d^\rho y}{dt^\rho} = L_f^\rho h(x) + L_g L_f^{\rho-1} h \cdot u$$

ρ-times diff. $\rightarrow$ u appears explicitly

Relative degree of linear systems

- Linear system

$$\dot{x} = Ax + bu$$
$$y = cx$$

is a special case of nonlinear system. $\rightarrow$

$$f(x) = Ax, \quad g(x) = b, \quad h(x) = cx$$

- Relative degree ρ of linear system

$$cb = cAb = cA^2b = \cdots = cA^{\rho-2}b = 0, \quad cA^{\rho-1}b \neq 0$$

$\rightarrow$ Difference of the orders of the denominator and numerator polynomials of the transfer function (Equivalent to the usual definition)

I/O exact linearization for SISO systems

- If the system has a relative degree ρ, the output can be differentiated ρ-times:

$$\frac{d^\rho y}{dt^\rho} = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) \cdot u$$

- A feedback

$$u = \frac{-L_f^\rho h(x) + v}{L_g L_f^{\rho-1} h(x)}$$

  linearizes I/O behavior as

$$\frac{d^\rho y}{dt^\rho} = v$$

- A further linear feedback of $y = h(x)$, $\dot{y} = L_f h(x)$,...,$d^{\rho-1}y/dt^\rho = L_f^{\rho-1} h(x)$ ($=$ nonlinear feedback of x) can perform pole assignment.
  Adding integrator or feedforward terms are also available.

Vector relative degree

Consider multi-input multi-output systems ($\ell \leq m$).

Definition:

The system has a vector relative degree $(\rho_1, \dots, \rho_\ell)$ at a point x_0, if there exists a neighborhood U_{x_0} of x_0 such that

$$
\begin{aligned}
&(L_{g_k} L_f^i h_j)(x) = 0, \quad j = 1, \dots, \ell, i = 0, \dots, \rho_j - 2, \\
&\qquad k = 1, \dots, m, {}^\forall x \in U_{x_0}
\end{aligned}
$$

$$
\text{rank} \underbrace{\begin{bmatrix} L_{g_1} L_f^{\rho_1 - 1} h_1(x_0) & \cdots & L_{g_m} L_f^{\rho_1 - 1} h_1(x_0) \\ & \vdots & \\ L_{g_1} L_f^{\rho_\ell - 1} h_\ell(x_0) & \cdots & L_{g_m} L_f^{\rho_\ell - 1} h_\ell(x_0) \end{bmatrix}}_{=G(x)} = \ell
$$

Then,

$$
\begin{pmatrix} \frac{d^{\rho_1} y_1}{dt^{\rho_1}} \\ \vdots \\ \frac{d^{\rho_\ell} y_\ell}{dt^{\rho_\ell}} \end{pmatrix} = \begin{pmatrix} L_f^{\rho_1} h_1(x) \\ \vdots \\ L_f^{\rho_\ell} h_\ell(x) \end{pmatrix} + G(x) u
$$

I/O linearization of MIMO system

Suppose that there exists a vector relative degree

- For example, by using a psuede inverse,

$$u = G^{\top}(x)(G(x)G^{\top}(x))^{-1}\left\{-\begin{pmatrix} L_f^{\rho_1}h_1(x) \\ \vdots \\ L_f^{\rho_\ell}h_\ell(x)\end{pmatrix} + v\right\}$$

  linearlizes the system as

$$\begin{pmatrix} \frac{d^{\rho_1}y_1}{dt^{\rho_1}} \\ \vdots \\ \frac{d^{\rho_\ell}y_\ell}{dt^{\rho_\ell}}\end{pmatrix} = v$$

- For the cases of square system ($m = \ell$), the simple matrix inverse $G(x)^{-1}$ can be utilized instead of the psuede inverse $G^{\top}(x)(G(x)G^{\top}(x))^{-1}$.

Cases with no vector relative degree

Cases with no vector relative degree include

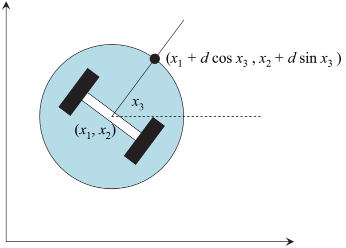
- cases when a relative degree can be recovered by a linear output coordinate transformation,
- cases when I/O linearization is available by adding a linear reference model system,
- cases when I/O linearization is available by a dynamic feedback,
- cases when I/O linearization is available by making a part of state space uncontrollable by partial inputs,
- and cases when I/O linearization is impossible.

Example — Two wheeled vehicle (1)

Two wheeled vehicle

$$\dot{x}_1 = u_1 \cos x_3$$
$$\dot{x}_2 = u_1 \sin x_3$$
$$\dot{x}_3 = u_2$$

(x_1, x_2) $\cdots$ Cartesian coordinate of the center of axle
x_3 $\cdots$ Heading angle
u_1 $\cdots$ Vehicle velocity (input 1)
u_2 $\cdots$ Yaw rate (input 2)



We consider an output which is the Cartensian coordinate of the front of the vehicle

$$y = \begin{pmatrix} x_1 + d\cos x_3 \\ x_2 + d\sin x_3 \end{pmatrix}$$

for the regularity of $G(x)$.

Example — Two wheeled vehicle (2)

Vector relative degree: $r = (1, 1)$
Time-derivative of the output:

$$\dot{y} = G(x)u = \begin{bmatrix} \cos x_3 & -d \sin x_3 \\ \sin x_3 & d \cos x_3 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

If $d \neq 0$, $G(x)$ is nonsingular.

Control law

$$u = \begin{bmatrix} \cos x_3 & \sin x_3 \\ -\sin x_3/d & \cos x_3/d \end{bmatrix} \begin{pmatrix} \dot{r}_x + k\{r_x - (x_1 + d\cos x_3)\} \\ \dot{r}_y + k\{r_y - (x_2 + d\sin x_3)\} \end{pmatrix}$$

(r_x, r_y) $\cdots$ Reference coordinate of the front of the car

Conclusion of exact I/O linearization

- Relative degree is defined as the times of time-derivative of the output where an input appears explicitly.
- By canceling nonlinear term and coefficient of the relative-degree-times of time-derivatives of the output, exact I/O linearization is realized.
- In the exact I/O linearization, a further feedback with pole assignment is often used for the stabilization.
- The order of the obtained dynamics representing I/O behavior is equal to the relative degree. Hidden dynamics will be referred in the next section.
- Exact I/O linearization of MIMO systems are also possible, under the assumption of the existence of vector relative degrees.

Normal Form

- The original system is n-dimensional, while the order of the I/O-behavior dynamics in the closed-loop system is ρ.
  What is the difference $n - \rho$?
- Coordinate transformation $\Phi(x)$: $x \to (z^\top, \xi^\top)^\top$

$$z_1 = h(x), z_2 = L_f h(x), \ldots, z_\rho = L_f^{\rho-1} h(x)$$

  The coordinate of ξ should be chosen to make the Jacobian matrix nonsingular.

Normal Form

$$\begin{aligned} y &= z_1 \\ \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_\rho &= L_f^\rho h(\Phi^{-1}(z,\xi)) + L_g L_f^{\rho-1} h(\Phi^{-1}(z,\xi)) \cdot u \\ \dot{\xi} &= \gamma(z,\xi) + \zeta(z,\xi)u \end{aligned}$$

In the case of SISO systems, making $\zeta(z,\xi) = 0$ is possible by choosing suitable coordinates.

Selection of Coordinate for $\zeta(\cdot) = 0$

The coordinate of ξ should be chosen to establish $\zeta(\cdot) = L_g\xi = 0$.

The number of the independent solutions of the partial differential equation:

$$L_g\xi = \frac{\partial \xi}{\partial x} g = 0$$

is $n - 1$. (Frobenius theorem, which will be described later)

The state of the I/O dynamics $z_1, \dots, z_{\rho-1}$ are also the solutions.
The coordinate of ξ should be chosen as $n - \rho$ functions from the solutions that are independent from $z_1, \dots, z_{\rho-1}$.

Zero dynamics

Suppose that the output is restricted to zero, i.e., $y \equiv 0$.
Time derivatives of y are also zero, so $z = 0$ holds. The input on the hypersurface $z = 0$ is

$$u = -\frac{L_g L_f^{\rho-1} h(\Phi^{-1}(0,\xi))}{L_f^{\rho} h(\Phi^{-1}(0,\xi))}$$

- By substituting it, we obtain $n - \rho$ dimensional zero dynamics

$$\dot{\xi} = \gamma(0,\xi) \underbrace{- \zeta(0,\xi) \frac{L_g L_f^{\rho-1} h(\Phi^{-1}(0,\xi))}{L_f^{\rho} h(\Phi^{-1}(0,\xi))}}_{\text{This part vanishes when } \zeta(z,\xi)=0}$$

- When y is not zero,
  - by giving the reference signal of y as a function of time, or
  - by considering an exo-system that generates the reference of y,

  we can define zero-error dynamics.

Zero dynamics of linear system

Example of linear case:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u \qquad \Rightarrow \quad G(s) = \frac{s-1}{s^2 - s - 1}$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

Exact I/O-linearization control law: $u = -x_1 - x_2 + v$
Closed-loop system:

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} v \qquad \Rightarrow \quad G(s) = \frac{\cancel{s-1}}{s\cancel{(s-1)}}$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

For linear systems, exact I/O linearization performs a pole assignment where

- transfer zeros are canceled ($\Rightarrow$ unobservable dynamics),
- and rest poles are assigned to zero.

Nonlinear non-minimum-phase system

Zero dynamics are invariant dynamics under feedbacks, which is similar to the fact that in linear cases transfer zeros are invariant under feedbacks.

- **Definition:** The system is called non-minimum phase, when its zero dynamics are unstable.
- Exact I/O linearization is not applicable to non-minimum phase systems.
  $\rightarrow$ It generates unstable internal dynamics which are unobservable.

Stability of cascaded system

- **Lemma:** Consider a system

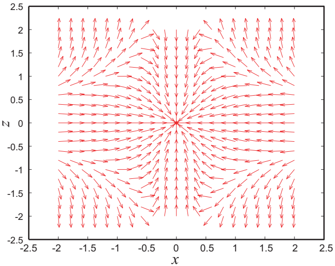
$$\dot{x} = f(x)$$
$$\dot{z} = g(z) + \gamma(x, z)x$$

where $\dot{x} = f(x)$ and $\dot{z} = g(z)$ are locally asymptotically stable, and $\gamma(x, z)$ is differentiable. Then the system is also locally asymptotically stable.

- However, even when $\dot{x} = f(x)$ and $\dot{z} = g(z)$ are globally asymptotically stable, The whole system may not be globally asymptotically stable.

[Ex.]

$$\dot{x} = -x$$
$$\dot{z} = -z + z^3x^2$$



Global asymptotical stability

Due to this fact, the combination of

[Globally asymptotically stable zero-dynamics]

+ [Exact linearization with stable I/O behavior]

does not mean global asymptotical stability.

[Example]

System:

$$\dot{x}_1 = x_2 + u$$
$$\dot{x}_2 = x_1 + x_1^2 x_2^3 + u$$
$$y = x_1$$

Zero dynamics:

$\dot{x}_2 = -x_2$ (GAS)

Control law: $u = -x_1 - x_2$

I/O behavior:

$\dot{x}_1 = -x_1$ (GAS)

However, the closed loop system becomes

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -x_2 + x_1^2 x_2^3$$

$\Rightarrow$ same as the previous slide

Peaking

Does making the error system fast solve this problem?

The answer is NO.
For the cases with relative degree 2 or higher, fast error dynamics may reduce the stability region.

Peaking

For the cases with relative degree 2 or higher, setting large absolute values of poles of error dynamics may cause large transient response.

Conclusion of zero dynamics

- When the relative degree is lower than the system dimension, exact I/O linearization generates "zero dynamics" which are unobservable.
- Zero dynamics are invariant under feedbacks.
- Exact I/O linearization cannot be applied to nonlinear non-minimum-phase systems. (It causes unstable internal dynamics.)
- Even when the zero dynamics are GAS, the controlled system with I/O linearization may not be GAS. Moreover, due to the peaking phenomenon, selecting poles cannot realize the enlargement of the stability region, generally.

Basic concept of exact state-space linearization

Exact I/O linearization is not applicable to nonlinear non-minimum-phase systems.
⇒ Minimum-phase property depends on the definition of the output function $h(x)$.

Problem

Find an output function $y = \lambda(x)$ such that the relative degree is n.

- Since $n - \rho = 0$, no zero dynamics exists for such an output.
- Therefore, exact I/O linearization for $\lambda(x)$ establishes linearization of the whole state-space. → Exact state-space linearization (厳密状態空間線形化)

Is the reverse proposition true?

State-space linearization and existence of $\lambda(x)$

- **Assumption:** There exist a state feedback $u = \alpha(x) + \beta(x)v$ ($\beta(x) \neq 0$) and a coordinate transformation $z = \Phi(x)$ such that the system can be transformed into a linear controllable canonical form

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 \\ -a_0 & \cdots & & -a_{n-1} \end{bmatrix} z + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} v$$

- For the output z_1, the closed-loop system has a relative degree n. Since feedback preserves the relative degree, the relative degree of the original system is also n for the output.

Theorem

An SISO input-affine nonlinear system can be converted into a controllable linear system by a state feedback, if and only if there exists an output function $\lambda(x)$ such that the relative degree coincides with the system dimension n.

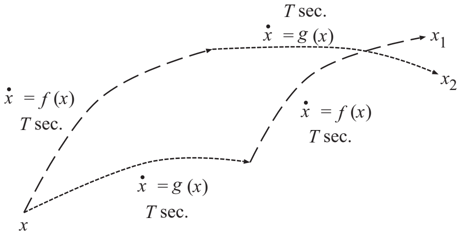
Lie bracket (1)

Definition of Lie bracket (リー括弧積)

$f(x)$, $g(x)$: $M \to TM$ (vector fields)

$$[f,g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \quad \text{(local-coordinate expression)}$$

A measure of non-commutability between two vector fields.



$$[f,g](x) = \lim_{T\to 0} \frac{1}{T^2}(x_1(x,T) - x_2(x,T))$$

Lie bracket (2)

- Various formulas $(a_1,\ a_2$: scalar constants$)$

$$
\begin{aligned}
&[f, g] = -[g, f] \\
&[a_1 f_1 + a_2 f_2, g] = a_1[f_1, g] + a_2[f_2, g] \\
&[f, a_1 g_1 + a_2 g_2] = a_1[f, g_1] + a_2[f, g_2] \\
&[f, [g, p]] + [g, [p, f]] + [p, [f, g]] = 0 \\
&\qquad\qquad\qquad\qquad \text{(Jacobi identity)} \\
&[\alpha f, \beta g] = \alpha\beta[f, g] + \alpha \cdot (L_f \beta) \cdot g - (L_g \alpha) \cdot \beta \cdot f \\
&L_{[f,g]} h = L_f L_g h - L_g L_f h \qquad \text{(IMPORTANT!)}
\end{aligned}
$$

Conditions of the output function

Conditions for the relative degree n

Condition 1 No input term appears until $(n-1)$-times derivative

$$
\begin{aligned}
(L_g\lambda)(x) &= 0 \\
(L_gL_f\lambda)(x) &= 0 \\
&\vdots \\
(L_gL_f^{n-2}\lambda)(x) &= 0
\end{aligned}
$$

Condition 2 An input term appears in the n-times derivative

$$(L_gL_f^{n-1}\lambda)(x) \neq 0$$

These conditions will be reinterpreted by using Lie bracket.
Formula:

$$L_{[f,g]}\lambda = L_fL_g\lambda - L_gL_f\lambda$$

Condition 1

We will express condition 1 by first-order partial differential equations as

$$
\begin{aligned}
& L_g \lambda = 0 \\
& L_g L_f \lambda = -L_{[f,g]}\lambda + L_f \underbrace{L_g \lambda}_{=0} = 0 \\
& L_g L_f^2 \lambda = -L_{[f,g]} L_f \lambda + L_f \underbrace{L_g L_f \lambda}_{=0} \\
& \qquad\quad = L_{[f,[f,g]]}\lambda - L_f \underbrace{L_{[f,g]}\lambda}_{=0} = 0 \\
& \qquad \vdots \\
& L_g L_f^{n-2} \lambda = (-1)^n L_{[f,[f \cdots [f,g] \cdots]]}\lambda = 0
\end{aligned}
$$

ad_f operator

Definition

$$\mathrm{ad}_f g = [f, g]$$

Multiple application:

$$\mathrm{ad}_f^k g = \underbrace{[f, [f \cdots [f, g]}_{k-\text{times}} \cdots]]$$

No action case:

$$\mathrm{ad}_f^0 g = g$$

Another expression of Condition 1

Another expression of Condition 1

$$
\begin{aligned}
(L_g \lambda)(x) &= 0 \\
(L_{\mathrm{ad}_f g} \lambda)(x) &= 0 \\
&\vdots \\
(L_{\mathrm{ad}_f^{n-2} g} \lambda)(x) &= 0
\end{aligned}
$$

Condition 2

By considering Condition 1, Condition 2 can be expressed by

$$
\begin{aligned}
(L_g L_f^{n-1}\lambda)(x) &= -(L_{[f,g]} L_f^{n-2}\lambda)(x) + (L_f \underbrace{L_g L_f^{n-2}\lambda}_{=0})(x) \\
&= L_{\mathrm{ad}_f^2 g} L_f^{n-3}\lambda - L_f L_{[f,g]} L_f^{n-3}\lambda \\
&= -L_{\mathrm{ad}_f^3 g} L_f^{n-4}\lambda + L_f L_{\mathrm{ad}_f^2 g} L_f^{n-4}\lambda - L_f L_g L_f^{n-2}\lambda + L_f^2 L_g L_f^{n-3}\lambda \\
&= \cdots = (-1)^{n-1} L_{\mathrm{ad}_f^{n-1} g}\lambda \neq 0
\end{aligned}
$$

Conditions for λ

Consequently, we obtain the following conditions:

Conditions for the output function

The necessary and sufficient condition that the output function $\lambda(x)$ should satisfy is

$$
\begin{aligned}
(L_g \lambda)(x) &= 0 \\
(L_{\mathrm{ad}_f g} \lambda)(x) &= 0 \\
(L_{\mathrm{ad}_f^2 g} \lambda)(x) &= 0 \\
&\vdots \\
(L_{\mathrm{ad}_f^{n-2} g} \lambda)(x) &= 0 \\
(L_{\mathrm{ad}_f^{n-1} g} \lambda)(x) &\neq 0
\end{aligned}
$$

Independence of vector fields

Consider vector fields

$$g, \mathrm{ad}_f g, \dots, \mathrm{ad}_f^{n-1} g$$

Reductio ad absurdum Suppose that $\mathrm{ad}_f^k g$ ($k \le n-1$) is linearly dependent upon g, $\mathrm{ad}_f g$,...,$\mathrm{ad}_f^{k-1} g$. Then, there exist coefficients c_i such that

$$\mathrm{ad}_f^k g(x) = c_0(x) g(x) + c_1(x)\mathrm{ad}_f g(x) + \cdots + c_{k-2}(x)\mathrm{ad}_f^{k-1} g(x)$$

Then,

$$\begin{aligned}\mathrm{ad}_f^{k+1} g(x) &= c_0(x)\mathrm{ad}_f g(x) + (L_f c_0)(x) g(x) + \\ &\quad \cdots + c_{k-3}(x)\mathrm{ad}_f^{k-1} g(x) + (L_f c_{k-3})(x)\mathrm{ad}_f^{k-2} g(x) \\ &\quad + c_{k-2}(x)\{c_0(x) g(x) + c_1(x)\mathrm{ad}_f g(x) + \cdots + c_{k-2}(x)\mathrm{ad}_f^{k-1} g(x)\} \\ &\quad + (L_f c_{k-2})(x)\mathrm{ad}_f^{k-1} g(x)\end{aligned}$$

holds. Hence, $\mathrm{ad}_f^{k+s} g(x)$ ($s = 1, 2, \dots$) are also linear dependent.
It contradicts the condition $L_{\mathrm{ad}_f^{n-1} g}\lambda(x) \neq 0$. Therefore, these vector fields are linearly independent under Conditions 1 and 2.

Necessary condition (A)

We obtain

Necessary condition of vector fields (A)

Vector fields

$$g, \mathrm{ad}_f g, \dots, \mathrm{ad}_f^{n-1} g$$

are linearly independent. (=Sufficient condition of local accessibility)

Integrability (1)

Condition 1 is equivalent to solving $(n-1)$ partial differential equations

$$
\begin{aligned}
(L_g \lambda)(x) &= 0 \\
(L_{\mathrm{ad}_f g} \lambda)(x) &= 0 \\
(L_{\mathrm{ad}_f^2 g} \lambda)(x) &= 0 \\
&\vdots \\
(L_{\mathrm{ad}_f^{n-2} g} \lambda)(x) &= 0
\end{aligned}
$$

We do not consider trivial solutions (constant solutions), which do not satisfy Condition 2.

$$
\left\langle \frac{\partial \lambda}{\partial x}, p(x) \right\rangle = \left[\frac{\partial \lambda}{\partial x_1}, \dots, \frac{\partial \lambda}{\partial x_n} \right] p(x) = 0, \quad p = g, \mathrm{ad}_f g, \dots, \mathrm{ad}_f^{n-2} g
$$

$\Rightarrow$ One form $\partial \lambda / \partial x$ is orthogonal to g, $\mathrm{ad}_f g, \dots, \mathrm{ad}_f^{n-2} g$.

Integrability (2)

In the n-dimensional space, there exists a nonzero one form that is orthogonal to $(n-1)$ vector fields g, $\mathrm{ad}_f g, \ldots, \mathrm{ad}_f^{n-2} g$.

$$\Downarrow$$

Let $\omega(x)$ be the one form. Can we generate a function $\lambda(x)$ with a scaling function $s(x)$ as $s(x)(\partial\lambda/\partial x) = \omega(x)$?

The answer is negative. Further condition is necessary to the integrability.
$\rightarrow$ Frobenius Theorem

Frobenius Theorem

- Consider q partial differential equations $L_{p_1}\lambda = 0, \dots, L_{p_q}\lambda = 0$ on $(x \in)\mathbb{R}^n$, where vector fields $p_1(x), \dots, p_q(x)$ are linearly independent.

Frobenius Theorem

These PDEs have $n-q$ independent solutions $\lambda_1(x), \dots, \lambda_{n-q}(x)$, if and only if the distribution

$$\Delta(x) = \mathrm{span}\{p_1(x), \dots, p_q(x)\}$$

is involutive.

- A distribution means a space spanned by some vector fields.
- **Definition:** Distribution $\Delta(x)$ is called "involutive", if

$$[\delta_1, \delta_2] \in \Delta, \quad {}^\forall\delta_1 \in \Delta, {}^\forall\delta_2 \in \Delta$$

holds.

Necessary condition (B)

A necessary condition of the existence of $\lambda(x)$:

Necessary condition for the vector fields (B)

Distribution

$$\text{span}\{g(x), \mathrm{ad}_f g, \ldots, \mathrm{ad}_f^{n-2} g\}$$

is involutive.

Necessary and sufficient condition of exact state-space linearization

Theorem

A necessary and sufficient condition of the exact state-space linearization is

- The distribution
$$\Delta_n = \mathsf{span}\{g, \mathrm{ad}_f g, \dots, \mathrm{ad}_f^{n-1} g\}$$
has a dimension n.
- The distribution
$$\Delta_{n-1} = \mathsf{span}\{g, \mathrm{ad}_f g, \dots, \mathrm{ad}_f^{n-2} g\}$$
is involutive.

- The necessity is obvious.
- The sufficiency can be shown by constructing a control law.

Construction of control law (1)

PDE $L_\delta \lambda(x) = 0$ ($\delta \in \Delta_{n-1}$) has one non-trivial solution $\lambda(x)$.

$$\overbrace{\frac{\partial \lambda}{\partial x}}^{\neq 0} \cdot \underbrace{[g, \mathrm{ad}_f g, \dots, \mathrm{ad}_f^{n-1} g]}_{\text{Regular (from conditions)}} = [0, \dots, 0, \underbrace{L_{\mathrm{ad}_f^{n-1} g} \lambda}_{\text{Therefore, this is nonzero}}]$$

Therefore, we can show

$$L_g \lambda = L_g L_f \lambda = \cdots = L_g L_f^{n-2} \lambda = 0$$
$$L_g L_f^{n-1} \lambda \neq 0$$

Hence the system has a relative degree n for the output $\lambda(x)$.

Construction of control law (2)

- **Coordinate transformation:**

$$\left.\begin{aligned} z_1 &= \lambda(x) \\ z_2 &= (L_f\lambda)(x) \\ &\vdots \\ z_n &= (L_f^{n-1}\lambda)(x) \end{aligned}\right. \quad \Rightarrow z = \Phi(x)$$

- **System with new coordinate:**

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & & 0 \end{bmatrix} z + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_f^n\lambda + L_gL_f^{n-1}\lambda \cdot u \end{pmatrix}$$

- **Control law:**

$$u = -\frac{L_f^n\lambda(x)}{L_gL_f^{n-1}\lambda(x)} + \frac{v}{L_gL_f^{n-1}\lambda(x)}$$

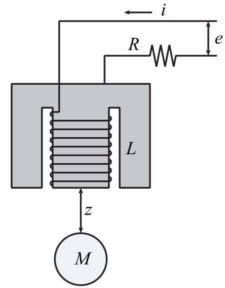
Example — Magnetic levitation system (1)

Magnetic levitation system:

$$M\ddot{z} = MG - K\cdot\left(\frac{i}{z+z_0}\right)^2$$

$$e = Ri + \frac{d}{dt}\{L(z)i\}$$

$$L(z) = \frac{2K}{z+z_0} + L_0$$



Magnetic levitation system

z —Gap between ball and magnet
i — Current
e — Voltage
M — Mass of ball
G — Acceleration of gravity

z_0 — Correction constant of gap
R — Electrical resistance
$L(z)$ — Inductance (function of z)
L_0 — Inductance on leakage flux
K $(= \mu_0 N^2 S/4)$ — Coefficient of force

Magnetic levitation system (2)

Equilibrium when $e = e_s$ **(constant)**:

$$\begin{pmatrix} z_s \\ \dot{z}_s \\ i_s \end{pmatrix} = \begin{pmatrix} \sqrt{K}e_s/(R\sqrt{MG}) - z_0 \\ 0 \\ e_s/R \end{pmatrix}$$

State: $x = (z - z_s, \dot{z}, i - i_s)^\top$
Input: $u = e - e_s$
State equation:

$$\dot{x} = \begin{pmatrix} x_2 \\ G - \dfrac{K(x_3 + i_s)^2}{M(x_1 + z_s + z_0)^2} \\ \phi(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1/L(x_1 + z_s) \end{pmatrix} u$$

$$\phi(x) = -\frac{1}{L(x_1 + z_s)} \left(Rx_3 + \frac{2Kx_2(x_3 + i_s)}{(x_1 + z_0 + z_s)^2} \right)$$

Magnetic levitation system (3)

$$g(x) = \begin{pmatrix} 0 \\ 0 \\ 1/L(x_1 + z_s) \end{pmatrix}$$

$$\mathrm{ad}_f g = [f, g] = \begin{pmatrix} 0 \\ \dfrac{2K(x_3 + i_s)}{M(x_1 + z_0 + z_s)^2 L(x_1 + z_s)} \\ \dfrac{R}{L(x_1 + z_s)^2} \end{pmatrix}$$

$$\mathrm{ad}_f^2 g = [f, [f, g]] = \begin{pmatrix} -\dfrac{2K(x_3 + i_s)}{M(x_1 + z_0 + z_s)^2 L(x_1 + z_s)} \\ * \\ * \end{pmatrix}$$

(Detail is omitted. The first element is nonzero.)

Condition (A) is satisfied.

$$\mathrm{rank}\Delta_3 = \mathrm{rank}\{f, [f, g], [f, [f, g]]\} = 3$$

Magnetic levitation system (4)

Condition (B) is also satisfied.

$$\Delta_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The first element of a vector field in Δ_2 is always zero.

$$[g, [f, g]] = \begin{pmatrix} 0 \\ \dfrac{2K}{M(x_1 + z_0 + z_s)^2 L(x_1 + x_s)^2} \\ 0 \end{pmatrix} \in \Delta_2$$

$\Rightarrow$ Δ_2 is involutive.

I/O linearization with an output $\lambda = x_1$ attains the state-space linearization of the system.

Conclusion of state-space linearization

- This method exactly linearizes a system via a state feedback and a coordinate transformation.
- A nonlinear system can be converted into a controllable linear system, if and only if there exists an output function with a relative degree n.
- It is relatively difficult to satisfy the condition, because it includes an integrability condition.
- However, most two-dimensional systems are exactly linearizable.
- Some higher-order systems originally have structures of linearizability.

Equilibrium

Equilibrium (平衡点)

For an autonomous system

$$\dot{x} = f(x)$$

an equilibrium (point) $x = x_0$ is a point such that $f(x_0) = 0$.

- Redefinition of the state coordinate where the origin $x = 0$ coincides with the equilibrium is often used. This procedure can be done without loss of generality.
- At the quilibrium, $\dot{x} = 0$ holds, i.e., the state is retained under the flow (the set of all orbits).
- In this section, we consider the stability properties of an equilibrium.

Definition of stability (1)

Boundedness (解の有界性)

A solution of a system $\dot{x} = f(x)$ starting from a neighborhood of its equilibrium $x = 0$ is bounded, if there exists a norm bound $K(x(0))$ such that $\|x(t)\| \le K(x(0))$ $(t \ge 0)$.

(Local) Stability → LS (局所安定性)

An equilibrium $x = 0$ of a system $\dot{x} = f(x)$ is (locally) stable, if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta(\epsilon) \Rightarrow \|x(t; x(0))\| < \epsilon, t \ge 0$$

- (Stable) $\subset$ (Bounded)
- For systems whose equilibrium $x = 0$ is stable, a solution starting from a neighborhood of the origin stays around the origin. For the case of limit cycle, the solutions are bounded but the origin is unstable.
- We call local stability 'Lyapunov stability.'
- The subject of the stability is an equilibrium, and is not a system.

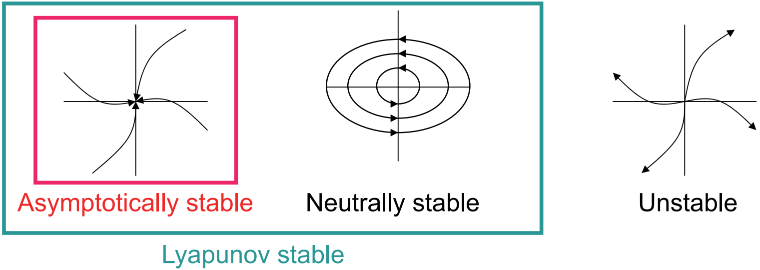
Definition of stability (2)

Attractiveness (吸引性)

If there exists a neighborhood U of the origin such that a solution starting from U satisfies $\|x(t;x(0))\| \to 0$ $(t \to \infty)$, the origin is called attractive. Then, U is called a domain of attraction.

(Local) Asymptotical Stability → LAS (局所漸近安定性)

An equilibrium $x = 0$ of a system $\dot{x} = f(x)$ is called (locally) asymptotically stable, if $x = 0$ is stable and attractive.



Definiton of stability (3)

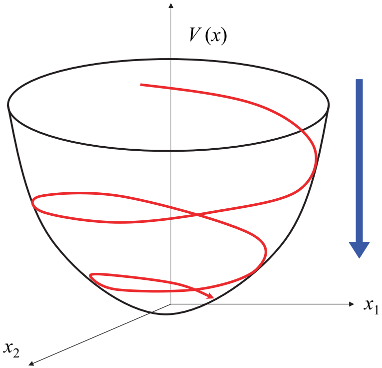
Global Stability → GS（大域的安定性）

An equilibrium $x = 0$ of a system $\dot{x} = f(x)$ is called globally stable, if $x = 0$ is stable and any solutions are bounded.

Global Asymptotical Stability → GAS（大域的漸近安定性）

An equilibrium $x = 0$ of a system $\dot{x} = f(x)$ is called globally asymptotically stable, if $x = 0$ is asymptotically stable, and its domain of attraction is the whole set of the state-space.

Concept of Lyapunov function



Candidate of **Lyapunov function**: $V(x)$
$\rightarrow$ A positive-definite function

Definition of positive-definite functions

- $V(0) = 0$
- $V(x) > 0, \quad x \neq 0$

$\Rightarrow$ Bowl-shaped function

[Ex.]

$$V(x) = x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 + x_2^2$$

If $V(x(t))$ decreases monotonically, $x(t)$ tends to the origin.

$\Rightarrow$ If $\dot{V}(x) < 0$ $(x \neq 0)$, then the origin is LAS.

Lyapunov theorem

Common condition: $V(x)$ is positive definite

LS: If

- $\dot{V} \leq 0$ around the origin,

the origin is (locally) stable.

LAS: If

- $\dot{V} < 0 \ (x \neq 0)$ around the origin,

the origin is (locally) asymptotically stable.

GS: If

- $\dot{V} \leq 0$, and
- $V(x)$ is radially unbounded,

then the origin is globally stable.

GAS: If

- $\dot{V} < 0 \ (x \neq 0)$, and
- $V(x)$ is radially unbounded,

then the origin is globally asymptotically stable.

Radial unboundedness (Definition)　(放射状に非有界条件)

$$V(x) \to \infty \quad (\|x\| \to \infty)$$

Lyapunov theorem gives a sufficient condition

These Lyapunov theorems give 'sufficient conditions.'

More specifically, we have to find the Lyapunov function by some means to show the stability of a stable nonlinear system. All positive-definite functions are not Lyapunov functions for a stable system.

However, there is a converse theorem in the sense of "existence theorem."

Converse Lyapunov theorem

Consider the system $\dot{x} = f(x)$ where $f(\cdot)$ is locally Lipschitz. Suppose that the origin of the system is globally asymptotically stable. Then, there exists a C^∞ Lyapunov function satisfying the radially-unbounded condition.

There are various types of 'converse Lyapunov theorems'. For example, see Y. Lin, E.D. Sontag, Y.Wang: "A Smooth Converse Lyapunov Theorem for Robust Stability", *SIAM J. Control Optim.*, **34**(1), 124–160, 1996.

Calculation of $\dot{V}$

We want to know the stability of the origin of

$$\dot{x} = f(x)$$

$\Rightarrow$ What is the role of $f(x)$ in the Lyapunov theory?

The vector field $f(x)$ is used in the calculation of $\dot{V}(x)$.

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial V(x)}{\partial x} f(x) (= L_f V(x))$$

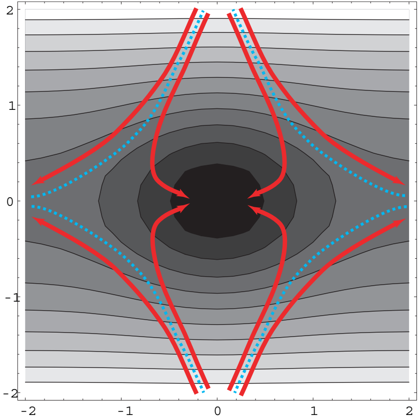
- Note that $\partial V / \partial x$ is a row vector in the local coordinate expression:

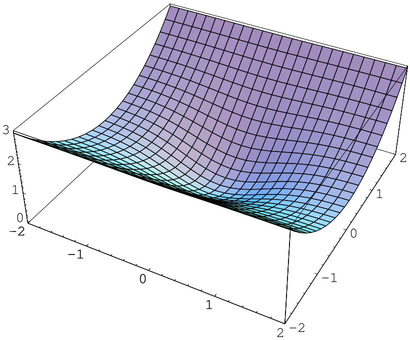
$$\frac{\partial V}{\partial x}(x) = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n} \right)$$

- L_f is the Lie derivative.

Radial unboundedness (1)

If the Lyapunov function is not ‘radially unbounded’, ...





- Locally asymptotically stable
- The origin is not globally asymptotically stable. $\Rightarrow$ The solutions outside of the separatrix, which is indicated light blue dotted curve, diverge.

Radial unboundedness (2)

Consider a locally Lipschitz autonomous system $\dot{x} = f(x)$. Suppose that $V(x)$ is positive definite, radially unbounded, differentiable, and its partial derivatives are continuous. If $\dot{V}(x)$ is negative definite, then

- Any sub-level set $S_a = \{x \mid V(x) \leq a\}$ $(a > 0)$ is compact, i.e., it is a bounded closed set.
- From the compactness and the continuity of $L_f V(x)$, $\dot{V}(x)$ is upper bounded on any level surface $\partial S_a = \{x \mid V(x) = a\}$, i.e.,

$$\dot{V}(x) \leq p(a) < 0, \quad {}^\forall x \in \partial S_a,\ a > 0.$$

Therefore,

$$\dot{V} \leq p(V) < 0$$

holds, and it is guaranteed that V converges to zero.

Without the radial unboundedness, the compactness is satisfied for only small a.

Weak Lyapunov function

- **Strong Lyapunov function**: Positive definite, and $\dot{V} < 0$ $(x \neq 0)$
  $\rightarrow \dot{V}$ is negative definite.
- **Weak Lyapunov function**: Positive definite, and $\dot{V} \leq 0$
  $\rightarrow \dot{V}$ is negative semi-definte.

A weak Lyapunov function guarantees only that the state converges to the set $\{x \mid \dot{V}(x) = 0\}$. (Barbalat's Lemma)

There exists a strong Lyapunov function around the origin that is asymptotically stable. (Converse Lyapunov theorem)
However, finding an explit form of a strong Lyapunov function is often difficult.

$$\Downarrow$$

Is it possible to ensure the asymptotical stability via a weak Lyapunov function with some conditions?

Barbalat's Lemma

- For a function $f(t)$, $\dot{f}(t) \to 0$ $(t \to \infty)$ does not imply that $f(t)$ has a limit at $t \to \infty$. **[Ex.]** $f(t) = \sin(\ln(t^2 + 1))$.
- Existence of a limit of a function $f(t)$ at $t \to \infty$ does not imply that $\dot{f}(t) \to 0$ $(t \to \infty)$. **[Ex.]** $f(t) = \sin(t^2)/\sqrt{t^2 + 1}$.

A weak Lyapunov function $V(x)$ has a lower bound $(V(x) \geq 0)$ and is a decreasing function $(\dot{V} \leq 0)$, and hence there exists a limit $V(x(+\infty))$. However it implies neither $V(x(+\infty)) = 0$ nor $\dot{V}(x(+\infty)) = 0$.

Barbalat's Lemma

If $f(t)$ has a finite limit as $t \to \infty$ and if $\dot{f}$ is uniformly continuous (or $\ddot{f}$ is bounded), then $\dot{f}(t) \to 0$ $(t \to \infty)$.

Application of Barbalat's Lemma to weak Lyapunov function

Suppose that there exists a weak Lyapunov function $V(x)$. Assume that $\partial V/\partial x$ and $f(x)$ are locally Lipschitz, then $\dot{V}(x(t)) = L_f V(x(t))$ is uniformly continuous, because the trajectory remains in a compact set $\{x \mid V(x) \leq V(x(0))\}$. Then, from Barbalat's lemma, we can conclude that $\dot{V} \to 0$ $(t \to \infty)$.

LaSalle's invariance principle

We consider a solution of $\dot{x} = f(x)$, where $f(x)$ is locally Lipschitz.

If for any initial state $x(0)$ included by a set Ω, $x(t) \in \Omega$ $(t > 0)$, then Ω is called a positively invariant set.

LaSalle's invariance principle (LaSalle の不変原理)

Let Ω be a positively invariant set. Suppose that any solution starting from Ω converges to a set $E(\subset \Omega)$. Then, any solution starting from Ω converges to M that is the maximal positively invariant set included in E.

In our case, Ω is often regarded as a compact set $\{x \mid V(x) \le a\}$ $(a > V(x(0)))$.

Asymptotical stability with weak Lyapunov function

Let $V(x)$ be a radially unbounded weak Lyapunov function. Suppose that $f(x)$ and the partial derivatives of $V(x)$ are locally Lipschitz. If the maximal positively invariant set included in $E = \{x \mid \dot{V}(x) = 0, V(x) \le a\}$ is $M = \{0\}$ for any $a = V(x(0)) > 0$, then the origin $x = 0$ is globally asymptotically stable.

LaSalle's invariance principle (cont.)

Proof: Set $\Omega = \{x \mid V(x) \le a\}$ $(a > 0)$, which is a compact positively-invariant set. From Barbalat's lemma, for any solution starting from Ω, $\dot{V}(x(t)) \to 0$ as $t \to \infty$. Therefore, any trajectory starting from a point in Ω converges to $E = \{x \mid \dot{V}(x) = 0, V(x) \le a\}$. From LaSalle's invariant principle, we can show that the state goes to $M = \{0\}$ as $t \to \infty$. The above discussion holds for any positive a $(= V(x(0)))$, and global Lyapunov stability of the origin is obvious. Consequently, the system is globally asymptotically stable.

Invariance principle — Example

Example

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x$$

$$V(x) = x^\top P x = x^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = x_1^2 + x_2^2$$

Time derivative of the Lyapuov function is

$$\dot{V}(x) = x^\top (PA + A^\top P) x = -2x_2^2$$

i.e., the state tends to a set $E = \{x \mid x_2 = 0\}$ as $t \to \infty$.

We will apply the invariance principle. If x stays in E, $\dot{x}_2 = 0$ should be satisfied. Only the origin satisfies $x \in E$ and $\dot{x}_2 = -x_1 - x_2 = 0$. Therefore, the maximal positively invariant set included in E is $\{0\}$, and the origin is globally asymptotically stable.

Conclusion of Lyapunov theorem

- Monotone-decrease property of a positive-definite Lyapunov function $V(x)$ guarantees the stability.
- Negative definite $\dot{V}(x)$ assures the asymptotical stability, while negative-semidefinite $\dot{V}(x)$ shows only stability.
- Radial unboundedness condition is necessary for global property.
- The invariance principle with a weak Lyapunov function helps us to show the asymptotical stability.

Concept of dissipativity

Storage function (ストレージ関数; 蓄積関数): $V(x)$

- Virtual energy function
- Generally positive semidefinite. However, in this lecture we mainly consider the positive-definite case.

Supply rate (供給率): $s(u, y)$

- Energy supplied from the environment par a unit time.
- A function of the input u and output y.

Rough concept of dissipativity (散逸性)

(Increase rate of storage function) $\leq$ (supply rate)

(RHS)$-$(LHS) indicates the dissipation of the virtual energy (≥ 0).

Definition of dissipativity

Definition of Dissipativity

A system is dissipative, if there exists a storage function $V(x)$ satisfying the dissipative inequality (散逸不等式)

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(u(t), y(t))dt$$

- $V(x)$: Storage function $\qquad V(0) = 0, \quad V(x) \geq 0$
- $s(\cdot)$: Supply rate

If V is differentiable, the dissipative inequality is equivalent to

$$\dot{V} \leq s(u, y)$$

$\Longrightarrow$ (Differential dissipative inequality)

A condition of dissipativity

Assumption

All points are reachable from the origin by choosing input u.

If the required supply

$$V^r(x(t_1)) \equiv \inf_{u,t_1} \left(\int_{t_0}^{t_1} s(u,y)\, d\tau \right), \quad x(t_0) = 0$$

is positive semidefinite, the system is dissipative for the supply rate $s(u,y)$ with the storage function $V_r(x)$.

As a matter of fact, a locally bounded available storage

$$V_a(x(t_0)) \equiv \sup_{u,t_1} \left(-\int_{t_0}^{t_1} s(u,y) dt \right)$$

also becomes a storage function. It is clear that $V_a(x)$ is positive-semidefinite (Consider the case $t_1 = t_0$). Moreover, all possible storage functions $V(x)$ satisfy $V_a(x) \leq V(x) \leq V_r(x)$.

A condition of dissipativity (Cont.)

Theorem

Suppose the reachable condition. Then, the dissipativity is equivalent to

$$\int_{t_0}^{t_1} s(u, y)dt \geq 0, \quad x(t_0) = 0, \ ^\forall u(\cdot)$$

Proof of necessity: It is obvious by the substitution of $x(t_0) = 0$ to the definition of dissipativity.

Proof of sufficiency: If the condition is satisfied, the required supply V_r is positive semidefinite. Hence, the system is dissipative for $s(u, y)$ with V_r.

Note that the above condition needs no information on the storage function $V(x)$. This condition show the existence of $V(x)$.

Various dissipativities

- Dissipativity for $s(u, y) = \gamma^2 \|u\|^2 - \|y\|^2$:
  $\Rightarrow$ Necessary and sufficient condition that the system has an L_2-gain from u to y that is lower than or equal to γ.
- Dissipativity for $s(u, y) = u^\top y$:
  $\Rightarrow$ Passivity
- Dissipativity for $s(u, y) = u^\top y - a\|u\|^2 - b\|y\|^2$:
  $\Rightarrow$ Generalization of passivity (circle criterion)

The passivity will be discussed in the next section.

Definition of passivity

Definition of Passivity (受動性)

Dissipativity for the supply rate $u^\top y$.

i.e., there exists a positive-semidefinite storage function such that

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} u^\top y \, dt.$$

- The numbers of input and output are same.
- If $V(x)$ is differentiable, it is equivalent to the differential passivity

$$\dot{V} \leq u^\top y$$

Examples of passive systems

- 2-terminal LCR circuit network, where the voltage is an input and the current is an output. We can regard the supplied power rate and the energy stored in the circuit as a supply rate and a storage function, respectively.
- A mechanical system with a positive-semidefinite Hamiltonian is a passive system, where the Hamiltonian, external forces, generalized velocities, and the work rate by the external forces are considered as a storage function, inputs, outputs, and a supply rate, respectively.
- A generalized Hamiltonian system

$$\dot{x} = (J - R)\left[\frac{\partial H}{\partial x}\right]^\top + g(x)u$$

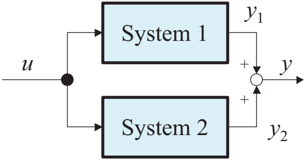
$$y = g(x)^\top \left[\frac{\partial H}{\partial x}\right]^\top$$

with a positive-semidefinite Hamiltonian $H(x)$ is also passive, where $J(x)$ is a skew symmetric matrix and $R(x)$ is positive definite.

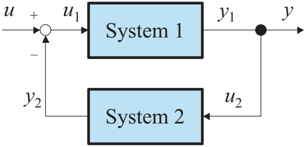
$$\dot{H} = -\left[\frac{\partial H}{\partial x}\right] R \left[\frac{\partial H}{\partial x}\right]^\top + y^\top u \leq u^\top y$$

Connections of two passive systems

- A Parallel connection of two passive systems is also passive.

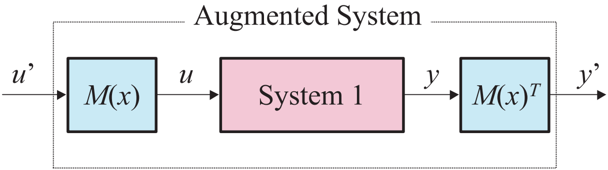


- A feedback connection of two passive systems is also passive.



Either subsystem has no direct feedthrough.

I/O transformation and passivity



An I/O transformation illustrated above preserves the passivity.

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} u^\top y dt = \int_{t_0}^{t_1} u'^\top M(x)^\top y dt$$
$$= \int_{t_0}^{t_1} u'^\top y' dt$$

IFP and OFP

OFP (Output Feedback Passivity)

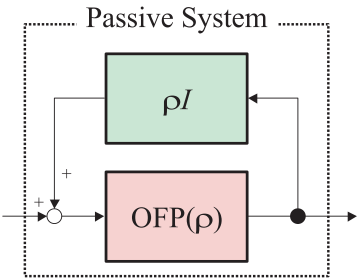
A system is called OFP(ρ), if it is dissipative for

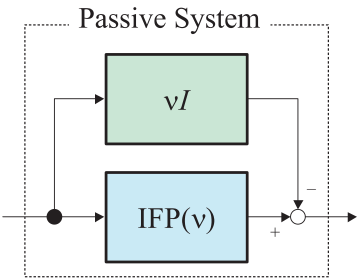
$$s(u,y) = u^\top y - \rho y^\top y$$

IFP (Input Feedback Passivity)

A system is called IFP(ν), if it is dissipative for

$$s(u,y) = u^\top y - \nu u^\top u$$

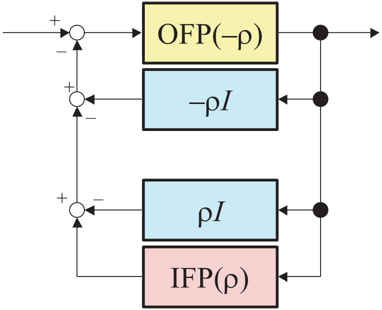




Properties of IFP/OFP systems

Assume that α is a positive constant.

- If a system Σ is OFP(ρ), $\alpha\Sigma$ is OFP(ρ/α).
- If a system Σ is IFP(ν), $\alpha\Sigma$ is IFP($\alpha\nu$).
- In the feedback connection, OFP($-\rho$) can be cancelled by IFP(ρ),



i.e., this inter-connected system is passive.

Stability of passive system (1)

Stability theory with a positive-semidefinite Lyapunov function

Suppose that there exists a Lyapunov function $V(x)$ such that

$$V(0) = 0, \quad V(x) \geq 0, \quad \dot{V} \leq 0.$$

Then, $E = \{x \mid V(x) = 0\}$ is a positively invariant set which includes the origin. If the origin is stable for the restricted dynamics on E, the origin for the original system is also stable.

Stability of passive system (2)

Stability properties of passive system

Consider a passive differentiable system $\dot{x} = f(x,u)$, $y = h(x,u)$ with a storage function $V(x)$.

1. If the storage function $V(x)$ is positive definite, the system with $u = 0$ is stable. In addition, if $V(x)$ is radially unbounded and positive definite, the system with $u = 0$ is globally stable.
2. If the system with $u = 0$ is zero-state detectable, the zero-input system is stable.
3. Suppose that the system has no direct feedthrough, i.e., the output function can be expressed as $y = h(x)$. Then, a feedback $u = -ky$ $(k > 0)$ asymptotically stabilizes the system, if and only if the closed-loop system is zero-state detectable.

Zero-state detectability

Consider a system,

$$\dot{x} = f(x,u), \qquad y = h(x,u).$$

Zero-State Detectability (**ZSD**; ゼロ状態可検出性):
The system is called zero-state detectable, if $y \equiv 0$ yields

$$x(t) \to 0 \quad (t \to \infty)$$

Zero-State Observability (**ZSO**; ゼロ状態可観測性):
The system is called zero-state observable, if $y \equiv 0$ yields

$$x(t) \equiv 0$$

For linear systems, ZSO coincides with the usual observability, and ZSD is equal to the usual detectability.

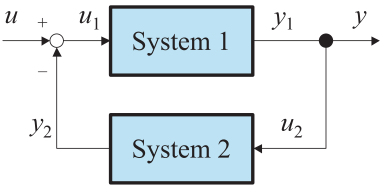
Proofs

1. For $u = 0$, the supply rate becomes zero. By regarding $V(x)$ as a Lyapunov function, $\dot{V}(x) \leq 0$.
2. Since $V(x) \geq 0$, for the point with $V(x) = 0$,

   $$0 \leq \dot{V}(x) \leq u^\top h(x, u), \quad \text{for } {}^\forall u$$

   holds. From the differentiability of $h(x, u)$, it can be decomposed as $h(x, u)$ $= h(x, 0) + \eta(x, u)u$. Therefore, when $V(x) = 0$, $u^\top h(x, 0) + u^\top \eta(x, u)u \geq 0$ for all u, which derives that $h(x, 0) = 0$. Hence, for a system $\dot{x} = f(x, 0)$, a maximal positively invariant set contained in $\{x \mid V(x) = 0\}$ is also included in $\{x \mid h(x, 0) = 0\}$. From the zero-state detectability, the state converges to the origin on the set $\{x \mid V(x) = 0\}$ Consequently, from the theorem, which is shown in previous slide, the origin is stable.
3. As the proof of 2, $y = h(x) = 0$ holds, when $V(x) = 0$. Because $\dot{V} \leq -kh(x)^\top h(x)$, the state converges to the set $\{x \mid h(x) = 0\}$. On the set, the input is zero. From ZSD of the system with zero input, the state tends to the origin as $t \to \infty$. The necessity can be also shown in a similar way.

Feedback connection of IFP and OFP systems



Suppose that Systems 1 and 2 with $u_1 = 0$, $u_2 = 0$ are ZSD.
Consider the case of $u = 0$.

- System 1 is dissipative for the supply rate

$$u_1^\top y_1 - \rho_1 y_1^\top y_1 - \nu_1 u_1^\top u_1$$

with a storage function $V_1(x_1)$.

- System 2 is dissipative for the supply rate

$$u_2^\top y_2 - \rho_2 y_2^\top y_2 - \nu_2 u_2^\top u_2$$

with a storage function $V_2(x_2)$.

1. If $\nu_1 + \rho_2 \geq 0$ and $\nu_2 + \rho_1 \geq 0$, then the closed-loop is stable.
2. If $\nu_1 + \rho_2 > 0$ and $\nu_2 + \rho_1 > 0$, then the closed-loop is asymptotically stable.

If V_1 and V_2 are positive definite and radially unbounded, The properties of 1. and 2. are "global."

Hint of Proof: Consider a Lyapunov function $V_1 + V_2$.

The case of static feedback

Assumptions: ■ System 1 with $u_1 = 0$ is ZSD.
■ V_1 is positive-definite and radially unbounded.

We regard a simple static feedback law $y_2 = K u_2$ as System 2, where K is a positive-definite matrix.

Let $\lambda_{\min}$ denote the minimal eigenvalue of K, and $\lambda_{\max}$ the maximal eigenvalue of K. The storage function of System 2 is zero, because it has no state variables. For $\rho_2 > 0$, ν_2 with $\lambda_{\min} - \rho_2 \lambda_{\max}^2 - \nu_2 > 0$, the following inequality holds:

$$u_2^\top y_2 - \rho_2 y_2^\top y_2 - \nu_2 u_2^\top u_2 \geq (\lambda_{\min} - \rho_2 \lambda_{\max}^2 - \nu_2) u_2^\top u_2 \geq 0.$$

For $^\exists \rho_2 > 0$, $^\exists \nu_2$ with $\lambda_{\min} - \rho_2 \lambda_{\max}^2 - \nu_2 > 0$, if $\nu_1 + \rho_2 > 0$ and $\nu_2 + \rho_1 > 0$, then the closed-loop system is GAS.

Especially,

When System 1 ($\nu_1 = 0$) is OFP(ρ_1), the closed-loop system is GAS for K with large eigenvalues. Moreover, for passive System 1 ($\rho_1 = 0$, $\nu_1 = 0$), any positive-definite K makes the closed-loop system GAS.

Theorem of Hill & Moylan

Input-affine system:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x) + j(x)u$$

Theorem of Hill & Moylan, 1976

This system is dissipative for a supply rate

$$s(u, y) = u^\top y - \rho y^\top y - \nu u^\top u$$

with a differential storage function $V(x)$, if and only if there exist a functions $q : \mathbb{R}^n \to \mathbb{R}^k$ with a suitable k and $W : \mathbb{R}^n \to \mathbb{R}^{k\times m}$ satisfying

$$L_f V = -\frac{1}{2} q(x)^\top q(x) - \rho h(x)^\top h(x)$$
$$L_g V(x) = h(x)^\top - 2\rho h(x)^\top j(x) - q^\top(x) W(x)$$
$$W(x)^\top W(x) = -2\nu I + j(x) + j(x)^\top - 2\rho j(x)^\top j(x)$$

Corollaries of theorem of Hill & Moylan (1)

Theorem of Hill & Moylan derives the followings:

Relative degree of IFP system

If a system is IFP(ν) for a positive ν, $j(x)$ is a regular matrix, i.e. the system has a vector relative degree zero.

Proof: Since $\rho = 0$, $j(x) + j(x)^\top = 2\nu I + W(x)^\top W(x)$ is regular.

Storage function of passive system (IMPORTANT)

If a system satisfying $j(x) = 0$ is passive with a storage function $V(x)$, then

$$L_f V \leq 0$$
$$L_g V(x) = h(x)^\top$$

holds, i.e the system is Lyapunov stable and the output function is explicitly restricted by the above equation.

Proof: Since $\rho = \nu = 0$, $W(x) = 0$ holds. From the theorem of Hill & Moylan, the proof is obvious.

Corollaries of theorem of Hill & Moylan (2)

Relative degree of passive system

If a system with $j(x) = 0$ is passive with a storage function $V(x)$, and if the output functions are independent, the system has a vector relative degree one around the origin, i.e., $(L_g h)(0)$ is regular.

Proof: Since $\partial V/\partial x(0) = 0$, we obtain

$$\frac{\partial h}{\partial x}(0) = \left(g^\top \frac{\partial^2 V}{\partial x^2} \right)(0)$$

We can express $\partial^2 V/\partial x^2(0)$ as $R^\top R$, because it is positive semidefinite. The independence of the output function means that $\partial h/\partial x$ has a full rank, and therefore $\operatorname{rank} Rg(0) = m$ holds. Consequently, we can conclude that

$$\operatorname{rank}\,(L_g h)(0) = \operatorname{rank}\,\{g(0)^\top R^\top R g(0)\} = m$$

Linear passive system (1)

We will apply the theorem of Hill & Moylan to linear systems.

Linear passive system with a positive-definite storage function

Suppose that a linear system $\dot{x} = Ax + Bu$, $y = Cx + Du$ is passive with a positive-definite quadratic storage function $V(x) = x^\top Px/2$ ($P > 0$). Then, there exists matrices L and W satisfying

$$
\begin{aligned}
PA + A^\top P &= -L^\top L \\
PB &= C^\top - L^\top W \\
W^\top W &= D + D^\top
\end{aligned}
$$

Espacially, for the case with $D = 0$,

$$
\begin{aligned}
PA + A^\top P &\leq 0 \\
PB &= C^\top
\end{aligned}
$$

holds.

Linear passive system (2)

Positive Real (正実性)

A linear square system $H(s) = C(sI - A)^{-1}B + D$ (minimum realization is assumed) is called positive real when the followings are satisfied:

1. $\mathrm{Re}\,(\lambda_i(A)) \leq 0, \; i = 1, \dots, n$
2. $H(j\omega) + H(-j\omega)^\top \geq 0, \; \forall \omega \notin \lambda_i(A)$
3. All eigenvalues s_i of A on the imaginary axis are simple, and their Residue matrices $\lim_{s \to s_i} (s - s_i)H(s)$ are Hermite and positive semidefinite.

Positive Real Lemma

A passive linear system with a positive-definite storage function is positive real. Conversely, a minimum realization of a positive real system $H(s)$ is passive with a positive-definite storage function.

Linear passive system (3)

Strictly Positive Real (強正実性)

A linear square system $H(s) = C(sI - A)^{-1}B + D$ (minimum realization is assumed) is called strictly positive real when the followings are satisfied:

1. $\mathrm{Re}\,(\lambda_i(A)) < 0,\ i = 1, \ldots, n$
2. $H(j\omega) + H(-j\omega)^\top > 0,\ \forall \omega \in \mathbb{R}$
3. $H(\infty) + H(\infty)^\top > 0$ or $\lim_{\omega \to \infty} \omega^{2(m-q)} \det[H(j\omega) + H(-j\omega)^\top] > 0$, where $q = \mathrm{rank}[H(\infty) + H(\infty)]$.

Kalman-Yakubovich-Popov Lemma (KYP Lemma)

A system is strictly positive real, if and only if there exist $P > 0$, L, W, and $\epsilon > 0$ such that

$$
\begin{aligned}
PA + A^\top P &= -L^\top L - \epsilon P \\
PB &= C^\top - L^\top W \\
W^\top W &= D + D^\top
\end{aligned}
$$

Especially, when $D = 0$, simplified relations $PA + A^\top P < 0$ and $PB = C^\top$ hold.

Linear passive system (4)

Suppose that for some positive constants ϵ_1, ϵ_2, a linear system is dissipative for

$$u^\top y - \epsilon_1 u^\top u - \epsilon_2 y^\top y$$

and ZSD with zero input. Then, the system is strictly positive real.

Proof: From the Hill-Moylan's theorem and the ZSD property, the linear system with zero input is asymptotically stable. The system is obviously IFP(ϵ_1), so $G(s) = C(sI - A)^\top B + D$ can be expressed by a parallel connection of $\epsilon_1 I$ and a passive system $G_k(s)$. Therefore,

$$G(j\omega) + G(j\omega)^\top = 2\epsilon_1 I + G_k(j\omega) + G_k(j\omega)^\top > 0$$

for $\omega \in \mathbb{R}$. Moreover, $G(j\infty) + G(j\infty)^\top$ is also positive definite.

Sector nonlinearity

Various different definitions of sector nonlinearities (セクタ型非線形性) exist. This lecture adopts the following definition.

A locally-Lipschitz static function $y_2 = \phi(u_2)$ satisfying

$$\left\| y_2 - \frac{\alpha + \beta}{2} u_2 \right\|^2 \leq \left\| \frac{\beta - \alpha}{2} u_2 \right\|^2$$

is called sector nonlinearity of (α, β).

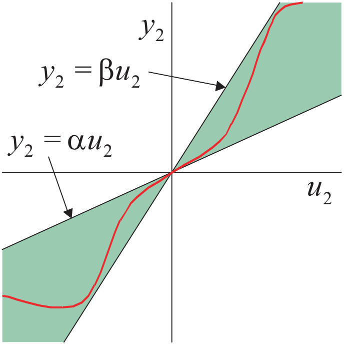
When $\beta = \infty$, by taking limit, it is defined as

$$u_2^\top y_2 \geq \alpha u_2^\top u_2$$

Scalar I/O case

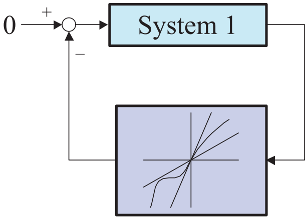
When the input and output are scalar, the sector nonlinearity is defined as

$$\alpha u_2^2 \leq u_2 y_2 \leq \beta u_2^2$$



Absolute stability

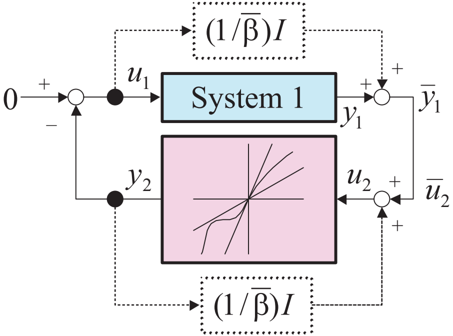
When feedback connected systems with System 1 and all sector nonlinearities of (α, β) are GAS, System 1 is called absolutely stable (絶対安定) for sector nonlinearities of (α, β).



Sufficient condition of absolute stability

Suppose that System 1 has no direct feedthrough, and is ZSD with zero input.

A sufficient condition of absolute stability for sector nonlinearities of (α, β) is that the parallel connected system of System 1 and a static gain function $(1/\bar{\beta})I$ is OFP($-k$) with a radially unbounded positive-definite differential storage function $V(x)$, where $k = \bar{\alpha}\bar{\beta}/(\bar{\beta} - \bar{\alpha})$, $\bar{\alpha} = \alpha - \epsilon_1$, $\bar{\beta} = \beta + \epsilon_2$ for $^\exists\epsilon_1 > 0$, and $^\exists\epsilon_2 > 0$. Let $\bar{\beta} = +\infty$, when $\beta = +\infty$.

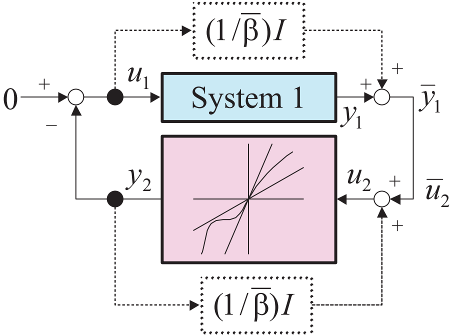


Proof of the sufficient condition

Proof From the OFP property, we get

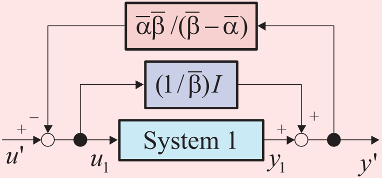
$$\dot{V} \leq \bar{y}_1^\top (u_1 + k\bar{y}_1) = -\bar{u}_2(y_2 - k\bar{u}_2)$$

By substituting $u_2 = \bar{u}_2 + y_2/\bar{\beta}$ into the definition of the sector nonlinearity, we obtain $\bar{u}_2(y_2 - k\bar{u}_2) > 0$ $(y_2 \neq 0)$. Hence, $\dot{V} < 0$ $(\bar{y}_1 \neq 0)$ holds. Note that $\bar{y}_1 = 0$ means $u_1 = 0$ and $y_1 = 0$. Since System 1 with zero input is ZSD, the feedback system is GAS.

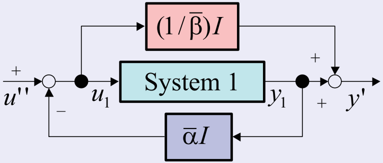


Another expression of the sufficient condition (1)

The sufficent condition is equivalent to the passivity of the following figure:



Moreover, since $u' = \beta/(\beta-\alpha)\cdot(u_1+\alpha y_1)$ and $y' = u_1/\beta + y_1$, it is also equivalent to the passivity of the figure below.



Another expression of the sufficient condition (2)

Since

$$
\begin{aligned}
u''^\top y &= (u_1 + \bar{\alpha} y_1)^\top (u_1/\bar{\beta} + y_1) \\
&= \frac{\bar{\alpha}+\bar{\beta}}{\bar{\beta}} \left\{ u_1^\top y_1 + \frac{1}{\bar{\alpha}+\bar{\beta}} u_1^\top u_1 + \frac{\bar{\alpha}\bar{\beta}}{\bar{\alpha}+\bar{\beta}} y_1^\top y_1 \right\}
\end{aligned}
$$

by choosing $\epsilon_1 = \epsilon_2$, we obtain the following lemma.

Suppose that $(\alpha+\beta)/\beta > 0$. Then, the sufficient condition for the absolute stability is equal to the dissipativity for the supply rate

$$
u^\top y + \frac{1}{\alpha+\beta} u^\top u + \left(\frac{\alpha\beta}{\alpha+\beta} - \bar{\epsilon} \right) y^\top y
$$

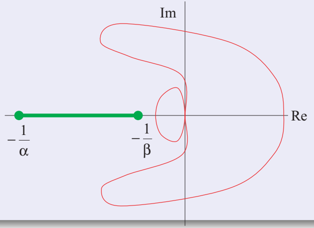
for some positive $\bar{\epsilon}$.

Stability margin of linear system (1)

Consider a Nyquist plot of an SISO strictly-proper linear system $G_0(s)$.
For simplicity, we assume that no pole of $G_0(s)$ is on the imaginary axis, and that it has p poles on the right-half plane of the complex plane.

Gain Margin (ゲイン余裕)

A system has a gain margin for (α, β), when the Nyquist plot turns around $-1/\kappa + j0$ $(\forall \kappa \in [\alpha, \beta])$ p-times counterclockwise.



Sector Margin (セクタ余裕)

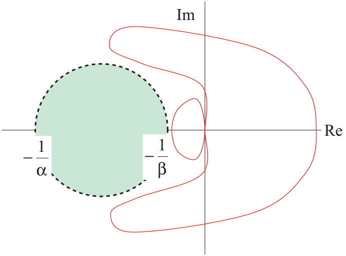
A system has a sector margin for (α, β), when it is absolute stable for the nonlinear sector of (α, β).

Stability margin of linear system (2)

Disc Margin (円盤余裕)

A system has a disk margin for $D(\alpha, \beta)$, when its Nyquist plot turns around the circle centered at $-\frac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)+j0$ with a radii $\frac{1}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$ p-times counterclockwise without contact. In this lecture, the contact is prohibited even for $G(j\infty)$. However, only when $\beta=+\infty$, we allow contacts between the Nyquist plot and the disk at the origin.

Let $D(\alpha, \beta)$ denote this disk.



Disk margin and positive realness

We assume $\beta > 0$.

Disk margin and positive realness

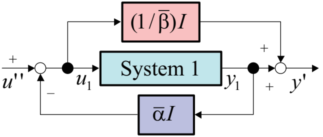
If $G_0(s)$ has a disk margin for $D(\alpha, \beta)$, if and only if

$$\bar{G}(s) = \frac{G_0(s) + (1/\bar{\beta})}{\bar{\alpha} G_0(s) + 1}$$

Same

is positive real.

System 1 is absolutely stable, if the following system is OFP(ϵ) for some positive ϵ:



When System 1 is linear with $G_0(s)$, the condition of the absolute stability is

$$\frac{L[y']}{L[u'']} = \frac{G_0(s) + (1/\bar{\beta})}{\bar{\alpha} G_0(s) + 1}$$

is passive.

Disk margin and positive realness (cont.)

- Since we only consider the minimum realization, the observability is satisfied.
- A positive-real linear system is always passive.
- When $\beta = +\infty$, we choose $\bar{\beta} = \infty$ also, i.e., $-1/\beta = -1/\bar{\beta} = 0$.

Consequently, we obtain the following result.

Disk criterion (circle criterion; 円盤条件)

If $G_0(s)$ has a disk margin for $D(\alpha, \beta)$, it has a sector margin for (α, β).

Generally, the converse of the above is not established.

$$\text{(Gain margin)} \supset \text{(Sector margin)} \supset \text{(Disk margin)}$$

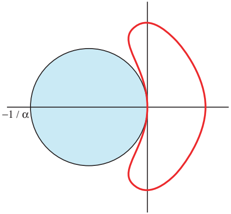
Disk margin and IFP/OFP

For controllable and observable linear system, the following facts are established.

OFP and disk margin

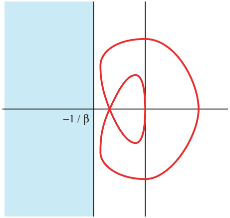
The followings are equivalent to each other.

1. A linear system is OFP$(-\alpha + \epsilon)$ with some positive ϵ.
2. The system has a disk margin for $D(\alpha, \infty)$.
3. The feedback connection of the system and all IFP(ν) $(\nu \geq \alpha)$ GAS linear system is GAS.



IFP and disk margin

A system is IFP$(-1/\beta + \epsilon)$ for some positive ϵ, if and only if it has a disk margin for $D(0, \beta)$.



A Review of analytical mechanics

Generalized position: $q = (q_1, \ldots, q_n)^\top$
External forces: $u = (u_1, \ldots, u_n)^\top$
Kinetic energy: $T(q, \dot{q})$
Potential energy: $W(q)$
Lagrangian: $L = T - W$

Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = u_i, \quad i = 1, \ldots, n$$

Vector expression:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right]^\top - \left[\frac{\partial L}{\partial q}\right]^\top = u$$

Usual mechanical system

Kinetic energy of usual mechanical systems can be expressed by quadratic form (二次形式)

$$T(q, \dot{q}) = \frac{1}{2} \dot{q}^\top M(q) \dot{q}$$

where $M(x)$ is a positive-definite inertia matrix (慣性行列).

Euler-Lagrange equation of usual mechanical system

$$M(q)\ddot{q} + c(q, \dot{q}) + g(q) = u$$

$$c(q, \dot{q}) = c_1(q, \dot{q}) + c_2(q, \dot{q})$$

$$c_1(q, \dot{q}) = \left[\frac{\partial (M(q)\dot{q})}{\partial q} \right] \dot{q} \quad \text{(Coriolis forces)},$$

$$c_2(q, \dot{q}) = -\frac{1}{2} \left[\frac{\partial (\dot{q}^\top M(q) \dot{q})}{\partial q} \right]^\top \quad \text{(centrifugal forces)}$$

$$g(q) = \left[\frac{\partial W}{\partial q} \right]^\top \quad \text{(gravity forces)}$$

Hamiltonian

Generalized momentums: $p = \left[\dfrac{\partial L}{\partial \dot{q}}\right]^\top$

Inverse transform: $\dot{q} = \phi(p, q)$

Hamiltonian

$$H(p, q) = \left[\dot{q}^\top p - L(q, \dot{q})\right]\Big|_{\dot{q}=\phi(p,q)}$$

For quadratic kinetic energy $T = \dot{q}^\top M(q)\dot{q}/2$ cases, ...

Generalized momentums: $p = M(q)\dot{q}$

Hamiltonian: $H = \dfrac{1}{2}p^\top M(q)^{-1}p + W(q)$

Hamiltonian expressed by q and $\dot{q}$:

$$H = \frac{1}{2}\dot{q}^\top M(q)\dot{q} + W(q) = T + W$$

Legendre transform

Infinitesimal displacement of L:
$$dL = \left[\frac{\partial L}{\partial \dot{q}}\right] d\dot{q} + \left[\frac{\partial L}{\partial q}\right] dq$$

Infinitesimal displacement of H expressed by p, $\dot{q}$, and L:
$$dH = \dot{q}^\top dp + p^\top d\dot{q} - dL$$

By substituting dL, from the definition of p, the term $p^\top d\dot{q}$ is canceled, and we obtain

$$dH = \dot{q}^\top dp - \left[\frac{\partial L}{\partial q}\right] dq$$

Finally, we get

$$\left[\frac{\partial H}{\partial p}\right] = \dot{q}^\top$$

$$\left[\frac{\partial H}{\partial q}\right] = -\left[\frac{\partial L}{\partial q}\right]$$

Partial derivative when H is a function of p, q

Partial derivative when L is a function of q, $\dot{q}$

Such a change of coordinate is called Legendre transformation.

Canonical equation

From Euler-Lagrange equation, we get

$$\dot{p} = \left[\frac{\partial L}{\partial q}\right]^\top + u$$

Finally, we obtain

Hamilton's canonical equation (Hamiltonian system)

Equation of motion expressed by p and q:

$$\dot{q} = \left[\frac{\partial H}{\partial p}\right]^\top$$

$$\dot{p} = -\left[\frac{\partial H}{\partial q}\right]^\top + u$$

Passivity of Hamiltonian system

Port-controlled Hamiltonian system

Hamiltonian system with I/O

$$\dot{q} = \left[\frac{\partial H}{\partial p}\right]^\top$$

$$\dot{p} = -\left[\frac{\partial H}{\partial q}\right]^\top + u$$

$$y = \left[\frac{\partial H}{\partial p}\right]^\top (= \dot{q})$$

$$\dot{H} = u^\top y$$

Port-controlled Hamiltonian systems is passive with storage function H.

Simple linear-feedback cases

We will consider control of port-Controlled Hamiltonian systems.

Assumptions

- The kinetic energy can be expressed by a quadratic form $T = \dot{q}^\top M(q)\dot{q}/2 = p^\top M(q)^{-1}p/2$ ($M(q) \geq M_0 > 0$).
- There exists a minimum value of the potential energy $W(q)$. Without loss of generarity, we set the minimum value zero.

Under a simple feedback $u = -ky = -k\dot{q}$, $\dot{H} = -ky^\top y = -k\dot{q}^\top \dot{q}$, i.e., $p \to 0$ ($t \to \infty$), which means that the closed-loop system is stabilized. Furthermore, if W is positive-definite and radially-unbounded with respect to q, and if $\partial W/\partial q \neq 0$ ($x \neq 0$), then from the invariance principle the closed-loop system is GAS.

Feedback $u = -k\dot{q}$ (D-control):
An equilibrium such that $W(q) = 0$ is asymptotically stabilized.

[Next slide]: We can change the potential W to a desired function.

Modification of the potential function

Desired potential function: $\overline{W}(q)$ (Positive-definite and radially-unbounded with respect to q, and $\partial\overline{W}/\partial q \neq 0$ ($x \neq 0$))
New Hamiltonian: $\bar{H}(p,q) = H(p,q) - W(q) + \overline{W}(q)$
$= T(p,q) + \overline{W}(q)$

$$\left[\frac{\partial H}{\partial p}\right]^\top = \left[\frac{\partial \bar{H}}{\partial p}\right]^\top, \quad \left[\frac{\partial H}{\partial q}\right]^\top = \left[\frac{\partial \bar{H}}{\partial q}\right]^\top + \left[\frac{\partial W}{\partial q}\right]^\top - \left[\frac{\partial \overline{W}}{\partial q}\right]^\top$$

By substituting these to the canonical equation, we get

$$\begin{aligned} \dot{q} &= \left[\frac{\partial \bar{H}}{\partial p}\right]^\top \\ \dot{p} &= -\left[\frac{\partial \bar{H}}{\partial q}\right]^\top - \left[\frac{\partial W}{\partial q}\right]^\top + \left[\frac{\partial \overline{W}}{\partial q}\right]^\top + u \\ y &= \left[\frac{\partial \bar{H}}{\partial p}\right]^\top (= \dot{q}) \end{aligned}$$

Canonical transformation

We will determine the feedback such that the closed-loop system becomes another canonical equation with the new Hamiltonian.

Input transformation (feedback): $u = g(q) - \left[\dfrac{\partial \overline{W}}{\partial q}\right]^\top + \bar{u}$

Obtained port-controlled Hamiltonian system:

$$\dot{q} = \left[\frac{\partial \bar{H}}{\partial p}\right]^\top \qquad y = \left[\frac{\partial \bar{H}}{\partial p}\right]^\top (= \dot{q})$$

$$\dot{p} = -\left[\frac{\partial \bar{H}}{\partial q}\right]^\top + \bar{u}$$

- A canonical transformation, which consists of the modification of Hamiltonian and feedback, derives another port-controlled Hamiltonian system.
- The transformed system is also passive, and asymptotically stabilizable by $\bar{u} = -ky$, where the state converges to the minimizer of the new potential, which can be designed freely.。

Quadratic case (1)

We consider the case where the new potential has a quadratic form.

$$\overline{W} = \frac{k_1}{2}(q - q_0)^\top (q - q_0)$$

where $k_1 > 0$ and q_0 is the reference point of q.
Input transformation:

$$u = g(q) - k_1(q - q_0) + \bar{u}$$

Under a further feedback $\bar{u} = -k_2 y = -k_2 \dot{q}$ ($k_2 > 0$), the equilibrium such that $\overline{W}(q) = 0$ (i.e., $q = q_0$) is GAS.

Quadratic case (2)

Finally obtained controller for the quadratic cases

$$u = g(q) - k_2\dot{q} - k_1(q - q_0)$$

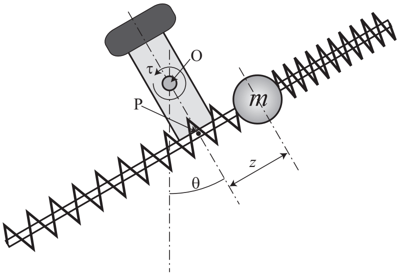
- $g(q)$: Cancelation of gravity.
- $-k_2\dot{q}$: Virtual viscosity friction: D(differential)-control.
- $-k_1(q - q_0)$: Virtual spring force: P(proportional)-control.

PD control with the cancelation of gravity makes the closed-loop system GAS.

Despite the nonlinearity,
combination of "cancelation of gravity" and "linear feedback"
globally asymptotically stabilizes the system.

$\Rightarrow$ Control utilizing the intrinsic property of the controlled object (dynamic-based control)

Example (1)



- A weight with mass m is sliding on a beam without frictions.
- The weight is supported by two springs, whose synthetic spring constant is K.
- The beam rotates centered at a point O. The length OP in the figure is L.
- The center of gravity of the beam without the moving weight coincides with O, and its moment of inertia is J.

- Let G denote the acceleration of gravity.
- The rotation axis is driven by a torque input τ.
- The controllability condition of the linearly approximated system holds, i.e., $LK \neq mG$.

Example (2)

- Let z denote the position of the moving weight from the point P, and θ be the angle of the beam.
- Let $q = (q_1, q_2)^\top = (\theta, z)^\top$ be the generalized-position vector, $\dot{q} = (\dot{q}_1, \dot{q}_2)^\top = (\dot{\theta}, \dot{z})$ the generalized-velocity vector, p the generalized-momentum vector.
- We regard $u = \tau$ as the input. The state vector is $x = (q^\top, \dot{q}^\top)^\top = (\theta, z, \dot{\theta}, \dot{z})^\top$, or $\tilde{x} = (q^\top, p^\top)^\top$.

Kinetic energy:

$$\begin{aligned} T &= \frac{J}{2}\dot{\theta}^2 + \frac{m}{2}\{(z^2 + L^2)\dot{\theta}^2 + 2L\dot{\theta}\dot{z} + \dot{z}^2\} \\ &= \frac{1}{2}\dot{q}^\top M(q)\dot{q} = \frac{1}{2}\dot{q}^\top \begin{bmatrix} J + m(L^2 + z^2) & mL \\ mL & m \end{bmatrix} \dot{q} \end{aligned}$$

Potential energy:

$$W = \frac{K}{2}z^2 + mg(L + y_m) = \frac{K}{2}z^2 + mg\{L(1 - \cos\theta) + z\sin\theta\}$$

Example (3)

Lagrangian: $L = T - W$
Euler-Lagrange equation:

$$M(q)\ddot{q} + c(q,\dot{q}) = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

$$c(q,\dot{q}) = \begin{pmatrix} 2mz\dot{\theta}\dot{z} + mG(z\cos\theta + L\sin\theta) \\ -mz\dot{\theta}^2 + Kz + mG\sin\theta \end{pmatrix}$$

Generalized momenta: $p = M(q)\dot{q}$
Hamiltonian: $H = \frac{1}{2}p^\top M(q)^{-1}p + W(q)$
Canonical equation:

$$\dot{q} = \frac{\partial H}{\partial p} (= \dot{q})$$

$$\dot{p} = -\frac{\partial H}{\partial q} + \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

Example (4)

Actually, the Hamiltonian is not positive-definite.

Modified Hamiltonian: $\bar{H} = H + \frac{k_1}{2} q_1^2$

$\bar{H}(\tilde{x})$ is positive definite, if $k_1 > m^2 G^2 / K$.

Input transformation: $u = -k_1 q_1 + v$
Transformed canonical equation:

$$\dot{q} = \frac{\partial \bar{H}}{\partial p} (= \dot{q})$$

$$\dot{p} = -\frac{\partial \bar{H}}{\partial q} + \begin{pmatrix} v \\ 0 \end{pmatrix}$$

$$y = \frac{\partial \bar{H}}{\partial p_1} = \dot{q}_1$$

This system is passive.

Example (5)

We will check the ZSD of this system. Suppose $y = 0$ and $u = 0$. Then, $q_1 = \theta = \theta_0$ (const.), $\dot{\theta} = 0$, and $\ddot{\theta} = 0$ hold, and the motion equations become

$$
\begin{aligned}
&mL\ddot{z} + mG(z\cos\theta_0 + L\sin\theta_0) + k_1\theta_0 = 0 \\
&m\ddot{z} + Kz + mG\sin\theta_0 = 0
\end{aligned}
$$

By eliminating $\ddot{z}$ from these equations, we get

$$
k_1\theta_0 = z(LK - mG\cos\theta_0)
$$

If $LK - mG\cos\theta_0 \neq 0$, z is a constant $z_0 (= k_1\theta_0/(LK - mG\cos\theta_0))$. By substituting $z = z_0$, $\dot{z} = 0$, and $\ddot{z} = 0$, and eliminating z_0, we obtain

$$
\begin{aligned}
&Kk_1\theta_0 + mG(KL - mG\cos\theta_0)\sin\theta_0 \\
&\qquad = Kk_1\theta_0 + mGKL\sin\theta_0 - \frac{m^2G^2}{2}\sin 2\theta_0 = 0
\end{aligned}
$$

For $k_1 > \max\{m^2G^2/K, mG(KL + mG)/4\}$, the sign of LHS corresponds to the sign of θ_0, and therefore $\theta = \theta_0 = 0$. Moreover, from the relation between θ_0 and z_0, $z = z_0 = 0$ should hold.

Example (6)

If $LK - mG\cos\theta_0 = 0$, then $\theta_0 = 0$ must be satisfied. This case appears only when $LK = mG$. (resonance condition)
However, it contradicts the assumption of the controllability of the linearly approximated system. Hence, we can show that $LK - mG\cos\theta_0 \neq 0$.

Consequently, we can conclude the zero-state detectability of the system for $k_1 > \max\{m^2G^2/K, mG(KL + mG)/4\}$. Then, the feedback $v = -k_2 y$ globally asymptotically stabilizes the origin.

Globally-asymptotically-stabilizing control-law

$$u = -k_1 q_1 - k_2 \dot{q}_1$$

where $k_1 > \max\{m^2G^2/K, mG(KL + mG)/4\}$ and $k_2 > 0$.

Passive output including positions (1)

Usual passivity-based control has an output that coincides with generalized velocity vector.
It may hinder the further utilization of the passive property, for example extention to passivity-based remote control.

It is useful to make the output include position information.

- **New generalized momentums**: $\tilde{p} = p + \lambda M(q)(q - q_0)$ $(\lambda > 0)$
- **New Hamiltonian**: $\tilde{H}(\tilde{p}) = \dfrac{1}{2}\tilde{p}^\top M(q)^{-1}\tilde{p}$

Passive output including positions (2)

Converted system

$$\dot{q} = \left[\frac{\partial H}{\partial p}\right]^\top = -\lambda(q - q_0) + M(q)^{-1}\tilde{p}$$

$$\dot{\tilde{p}} = -\left[\frac{\partial H}{\partial q}\right]^\top + \lambda\{\dot{M}(q,\tilde{p}) + p\} + u + F_{\mathrm{ext}}$$

- Blue term: Additional dumping.
- By making the red terms equal to $J(\tilde{p}, q)\left[\frac{\partial \tilde{H}}{\partial \tilde{p}}\right]^\top$, a new passivity-based control law can be derived, where $J(\tilde{p}, q)$ is a skew symmetric matrix.
- The closed-loop system is a cascaded connection of a generalized Hamiltonian system and a GAS system. Usually, it is assumed that $\|M(q)^{-1}\|$ is bounded to avoid finite-time divergence.
- The new output is $\tilde{y} = \left[\frac{\partial \tilde{H}}{\partial \tilde{p}}\right]^\top = \dot{q} + \lambda(q - q_0)$.

  $\Longrightarrow$ Useful for position synchronization in bilateral manipulator system.

Necessary condition of Lyapunov function of the closed-loop system (1)

Assumption

For a system

$$\dot{x} = f(x) + g(x)u$$

a stabilizing static state feedback $u = \alpha(x)$ is designed with a radially unbounded Lyapunov function $V(x)$, such that the closed loop system

$$\dot{x} = \tilde{f}(x) = f(x) + g(x)\alpha(x)$$

is GAS with $V(x)$.

What is the condition that $V(x)$ should satisfy?

Necessary condition of Lyapunov function of the closed-loop system (2)

Since

$$\dot{V} = \left[\frac{\partial V}{\partial x}\right]\{f(x) + g(x)\alpha(x)\}$$
$$= L_f V(x) + (L_g V(x))\alpha(x) < 0, \quad (x \neq 0),$$

at a glance, it seems that by choosing α large with the opposite sign of $L_g V$, we can always make $\dot{V}$ negative.

However, At a point such that $L_g V(x) = 0$, the input $u = \alpha(x)$ is **not** directly effective on $\dot{V}$.
$\Rightarrow$ At such points, $L_f V < 0$ $(x \neq 0)$ is necessary.

Necessary condition of Lyapunov function

At a point such that $L_g V(x) = 0$ and $x \neq 0$, the inequality $L_f V(x) < 0$ holds.

This condition is independent from the choice of the control law $\alpha(x)$.

Control Lyapunov function

Control Lyapunov function; clf

A function $V(x)$ is called a control Lyapnov function of a system $\dot{x} = f(x) + g(x)u$, when

- $V(x)$ is smooth radially-unbounded positive-definite function, and
- $L_f V(x) < 0$ holds at a point such that $L_g V(x) = 0$ and $x \neq 0$.

This is a **necessary condition** for the Lyapunov function of the closed-loop system.

In the following slides,...

We will show that the existence of a clf yields the global asymptotical stabilizability by a state feedback $u = \alpha_s(x)$, where we allow the local unboundedness around the origin.

Sontag-type control law

If a clf $V(x)$ exists,
Sontag-type control law

$$u = \alpha_s(x) = \begin{cases} -\dfrac{L_fV + \sqrt{(L_fV)^2 + (L_gV(L_gV)^\top)^2}}{L_gV(L_gV)^\top}(L_gV)^\top, & L_gV \neq 0 \\ 0, & L_gV = 0 \end{cases}$$

globally asymptotically stabilizes the system.

For the purpose of the asymptotical stabilization, only finding a clf achieves the objective, instead of the direct design of control laws.

Global asymptotical stabilization by Sontag-type controller

We will confirm that the Sontag-type controller globally asymptotically stabilizes the system.

We will calculate the time-derivative of the clf under the Sontag-type controller.

- When $L_gV \neq 0$,

$$\begin{aligned}\dot{V} &= L_fV + L_gV\alpha_s(x) \\ &= L_fV - \left\{L_fV + \sqrt{(L_fV)^2 + (L_gV(L_gV)^\top)^2}\right\} \\ &= -\sqrt{(L_fV)^2 + (L_gV(L_gV)^\top)^2} < 0\end{aligned}$$

- When $L_gV = 0$ and $x \neq 0$,

$$\dot{V} = L_fV < 0$$

Hence, $\dot{V}$ is negative definite. $\Rightarrow$ Consequently, the closed-loop system is GAS.

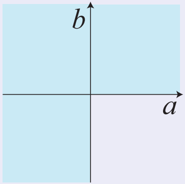
Continuity of Sontag-type controller

Is the Sontag-type controller $\alpha_s(x)$ locally bounded around the hyper-surface $L_g V = 0$?

Lemma: A function

$$\phi(a,b) = \begin{cases} 0, & \text{if } b = 0 \text{ and } a < 0 \\ -\dfrac{a + \sqrt{a^2 + b^2}}{b}, & \text{elsewhere} \end{cases}$$

is real analytic on $S = \{(a,b) \in \mathbb{R}^2 \mid b > 0 \text{ or } a < 0\}$.



Proof: Consider $F(a,b,p) = bp^2 - 2ap - b = 0$, which is a quadratic equation with respect to p. Its solution on S is $p = \phi(a,b)$, even when $b = 0$ and $a < 0$. Because

$$\frac{\partial F}{\partial p}(a, b, \phi(a,b)) = 2\sqrt{a^2 + b^2} \neq 0, \ (a,b) \in S$$

we can conclude that $\phi(a,b)$ is real analytic, by the implicit function theorem.

Continuity of Sontag-type controller (cont.)

Real analyticity except around the origin

For a clf $V(x)$, the Sontag-type controller $\alpha_s(x)$ is real analytic except around the origin.

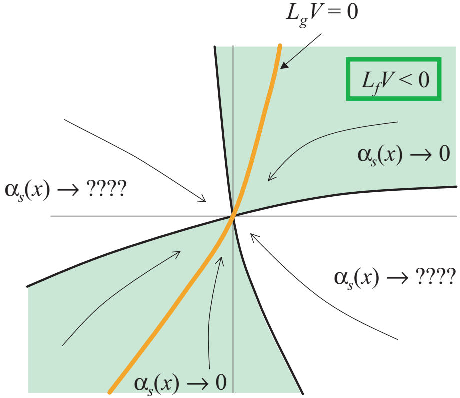
Brief proof: By substituting $a = L_fV$ and $b = L_gV(L_gV)^\top$ into the lemma on the previous slide, we get

$$\alpha_s(x) = \begin{cases} 0, & x = 0 \\ -\phi(L_fV, L_gV(L_gV)^\top)(L_gV)^\top, & x \neq 0 \end{cases}$$

Obviously, $\alpha_s(x)$ is real analytic except around the origin.

Sontag-type controller around the origin

Is the Sontag-type control law continuous around the origin?
Is the Sontag-type control law locally bounded around the origin?



It is not guaranteed that $\alpha_s(x)$ converges to zero when x approaches to the origin from the area of $L_fV > 0$. It depends on the ordres of L_gV and L_fV with respect to x around the origin.

$\Rightarrow$ Sontag's controller α_s may diverges around the origin.

Small control property

Small Control Property; scp

(Definition): A clf $V(x)$ has a small control property, if there exists a continuos control law $\alpha_c(x)$, defined around the origin, such that $\alpha_c(0) = 0$ and

$$L_f V(x) + L_g V(x)\alpha_c(x) < 0, \quad {}^\forall x \neq 0$$

Continuity of Sontag-type controller for a clf with scp

If a clf $V(x)$ has a scp, i.e., if there is a continuous asymptotically-stabilizing control law around the origin with a Lyapunov function, Sontag-type controller is also continuous.

The proof is on the next slide.

Small control property (cont.)

Proof: Since the control law is real analytic except around the origin, we will show the continuity only in the neighborhood of the origin.
From

$$|L_fV| \leq \|L_gV\| \cdot \|\alpha_c\|, \quad L_fV \geq 0$$

we obtain

$$\|\alpha_s\| \leq \|\alpha_c\| + \sqrt{\|\alpha_c\|^2 + \|L_gV\|^2}, \quad L_fV \geq 0$$

On the other hand, it is obvious that

$$\|\alpha_s\| \leq \|L_gV\|, \quad L_fV \leq 0$$

Because α_c and L_gV are continuous, $\alpha_s \to 0$ $(x \to 0)$. Therefore, α_s is also continuous.

If we can obtain a clf $V(x)$ with scp, the Sontag-type controller can globally asymptotically stabilizes the system.

Stabilizability and clf

We can get a necessary and sufficient condition of the global asymptotical stabilizability.

Theorem

A smooth system $\dot{x} = f(x) + g(x)u$ ($f(0) = 0$) can be globally asymptotically stabilized by a continuous state feedback, which vanishes at the origin, if and only if there exists a clf with scp in a weak sense, where the clf must be infinity-times differentiable, but does not have to be smooth.

The condition of clf is weakened due to the limitation of the converse Lyapunov theorem.

Finally, we get the following result.

For a system that is globally asymptotically stabilizable by a state feedback having small value near the origin, there exists a C^{∞}-class clf $V(x)$ with scp, and the Sontag-type controller, which achieves the global asymptotical stabilization, is continuous.

Notations for ISS (1)

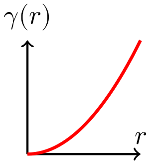
$\mathbb{R}^+$: Class of non-negative real numbers.

Class-$\mathcal{K}$

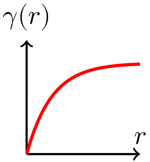
A strictly increasing continuous function $\gamma\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma(0) = 0$ is called to be a class-$\mathcal{K}$ function.

Class-$\mathcal{K}_\infty$

A class-$\mathcal{K}$ function $\gamma\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma(r) \to \infty$ $(r \to \infty)$ is called to be a class-$\mathcal{K}_\infty$ function.



Class-$\mathcal{K}_\infty$ function



Class-$\mathcal{K}$ function but not class-$\mathcal{K}_\infty$ function

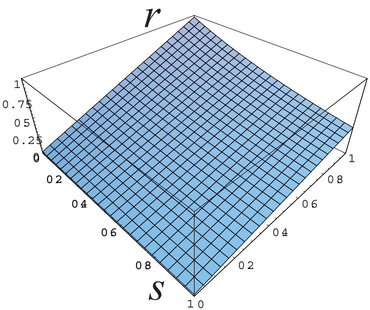
A class-$\mathcal{K}_\infty$ function γ has an inverse map $\gamma^{-1}\colon \mathbb{R}^+ \to \mathbb{R}^+$.

Notations for ISS (2)

Class-$\mathcal{KL}$

A continuous function β: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions is called a class-$\mathcal{KL}$ function:

1. for any fixed s, the function $\beta(\cdot, s)$ belongs to class-$\mathcal{K}$,
2. and for any fixed r, the function $\beta(r, \cdot)$ is a descresing function such that $\beta(r, s) \to 0$ $(s \to \infty)$.



Definition of ISS

Definition of input-to-state stability (ISS)

Cosider a nonlinear system

$$\dot{x} = f(x,t) + g_w(x,t)w,$$

where $f(0,t) = 0$. The system is called to be **input-to-state stable (ISS)**, if for any continuous $w(t)$ $(t \geq 0)$ and for any initial state $x(0)$ the solution $x(t)$ exists and there exist a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class-$\mathcal{K}$ function $\chi(\cdot)$ such that

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \chi\left(\sup_{0\leq\tau\leq t} \|w(\tau)\|\right), \quad {}^\forall t > 0.$$

- The function $\beta(\cdot,\cdot)$ in the above inequality represents an effect, which decays over time, of the initial condition.
- The existence of the function $\chi(\cdot)$ means that if the disturbance $w(t)$ is bounded, its effect is also bounded.

ISS for linear systems

The solition of a linear time-invariant (LTI) system $\dot{x} = Ax + bw$ is

$$x(t) = \exp(tA)x(0) + \int_0^t \exp((t-\tau)A)u(\tau)\,d\tau.$$

Therefore, the ISS condition of the LTI systems becomes

$$\|x(t)\| \leq \exp\left\{\left(\max_i \mathrm{Re}[\lambda_i(A)]\right)t\right\} + \int_0^\infty \|\exp(\tau A)B\|\,d\tau \cdot \left(\sup_{0\leq\tau\leq t}\|w(\tau)\|\right).$$

Hence, if $\mathrm{Re}[\lambda_i(A)] < 0$, the system is ISS. Furthermore, the converse is also true.

LTI system $\dot{x} = Ax + bw$ is ISS, if and only if all eigenvalues of A have negative real parts.

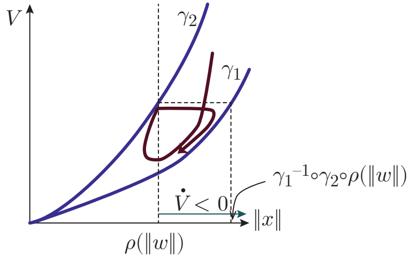
This ISS condition coincides with that of the usual stability of LTI systems.

A necessary and sufficent condition of ISS

Necessary and sufficent condition of ISS (Sontag & Wang 1995)

Nonlinear system $\dot{x} = f(x,t) + g_w(x,t)w$ with $f(0,t) = 0$ is ISS if and only if there exists a function $V(x,t)$: $\mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$, class-$\mathcal{K}_\infty$ functions $\gamma_1(\cdot)$, $\gamma_2(\cdot)$, and $\rho(\cdot)$, and a class-$\mathcal{K}$ function $\gamma_3(\cdot)$ such that

$$\gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|),$$
$$\dot{V} = (L_f V)(x,t) + (L_{g_w} V)(x,t)w \le -\gamma_3(\|x\|) \quad \text{for } \|x\| \le \rho(\|w\|).$$



Another necessary and sufficent condition of ISS

We will only consider the time-invariant systems. Then, the condition $\gamma_1(\|x\|) \leq V(x) \leq \gamma_2(\|x\|)$ means that $V(x)$ is a radially unbounded positive definite function.

Necessary and sufficent condition of ISS (Sontag & Wang 1996)

A nonlinear system $\dot{x} = f(x) + g_w(x)w$ with $f(0) = 0$ is ISS, if and only if there exists a radially unbounded positive definite function $V(x)$ such that

$$\dot{V} \leq -a(\|x\|) + b(\|w\|),$$

where $a(\cdot)$ and $b(\cdot)$ are some class-$\mathcal{K}_\infty$ functions.

For any fixed $\|w\|$, $\dot{V} < 0$ holds in the domain $\|x\| \geq a^{-1} \circ b(\|w\|) + \epsilon(\|w\|)$, where $\epsilon(\|w\|)$ is a small positive real number.

Integral ISS and integral–integral criterion

- **Max–Max criterion** (ISS)

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \chi(\|w(\cdot)\|_{2,\infty})$$

- **Integral–Max criterion** (Integral ISS; iISS)

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \int_0^t \chi(\|w(\tau)\|)\, d\tau$$

- **Integral–Integral criterion**

$$\int_0^\infty \alpha(\|x(\tau)\|)\, d\tau \leq \beta(\|x(0)\|, t) + \int_0^t \chi(\|w(\tau)\|)\, d\tau$$

The Integral–Integral criterion is equivalent to the Max–Max criterion (ISS).

Any ISS systems are iISS. The converse is not true.

Input-to-state stabilization

Consider the following nonlinear system with input u and disturbance w:

$$\dot{x} = f(x) + g_w(x)w + g(x)u \qquad (f(0) = 0).$$

Objective of input-to-state stabilization

Find a feedback law $u = \alpha(x)$ such that the closed-loop system

$$\dot{x} = \tilde{f}(x) + g_w(x)w = \{f(x) + g(x)\alpha(x)\} + g_w(x)w$$

is ISS.

ISS clf

Definition of ISS control Lyapunov function (ISS-clf)

A smooth radially unbounded positive-definite function $V(x)$ is called an ISS-clf, if there exists a class-$\mathcal{K}_\infty$ function $\rho(\cdot)$ such that

$$\inf_{u\in\mathbb{R}^m}\{L_fV(x)+L_gV(x)u+L_{g_w}V(x)w\}<0$$

for any (x,w) satisfying $\|x\|\geq\rho(\|w\|)$ and $x\neq 0$.

$\Longrightarrow$ There exists an input fulfilling the ISS condition.

Necessary and sufficient condition for ISS clf

Necessary and sufficient condition for ISS clf

A smooth radially unbounded positive-definite function $V(x)$ becomes an ISS-clf, if and only if there exists a class-$\mathcal{K}_\infty$ function $\rho(\cdot)$ such that

$$L_f V(x) + \|L_{g_w} V(x)\| \rho^{-1}(\|x\|) < 0$$

for x satisfyng $L_g V(x) = 0$ and $x \neq 0$

Proof: **[Necessity]** Assume that $L_g V(x) = 0$. The original inequality must be satisfied for $w = \{\rho^{-1}(\|x\|)/\|L_{g_w} V\|\}(L_{g_w} V)^\top$, and it derives the condition.
[Sufficiency] Suppose that the above condition is satisfied. For (x, w) such that $\|x\| \geq \rho(\|w\|)$ and $x \neq 0$, the following inequality holds:

$$\begin{aligned} \inf_u\{L_f V + L_g V \cdot u + L_{g_w} V \cdot w\} &\leq \inf_u\{L_f V + L_g V \cdot u + \|L_{g_w} V\| \cdot \|w\|\} \\ &\leq \inf_u\{L_f V + L_g V \cdot u + \|L_{g_w} V\| \rho^{-1}(\|x\|)\} = \begin{cases} < 0 & (L_g V(x) = 0) \\ = -\infty & (L_g V(x) \neq 0). \end{cases} \end{aligned}$$

Hence, the original inequality is satisfied.

Yet another necessary and sufficient condition of ISS

The condition on the previous slide derives a yet another ecessary and sufficient condition of ISS for no input cases. Suppose that $g(x) = 0$, and apply the condition on the previous slide to this case.

Necessary and sufficient condition of ISS

A nonlinear system $\dot{x} = f(x) + g_w(x)w$ with $f(0) = 0$ is ISS, if and only if there exists a radially unbounded positive definite function $V(x)$ and a class-$\mathcal{K}_\infty$ function such that

$$L_f V(x) + \|L_{g_w} V(x)\| \rho^{-1}(\|x\|) < 0 \quad (x \neq 0).$$

Necessary and sufficient condition for ISS stabilization (1)

Necessary and sufficient condition for ISS stabilization

An analytic system $\dot{x} = f(x) + g(x)u + g_w(x)w$ is ISS stabilizable, if and only if there exists an ISS-clf with scp (small control property).

Proof: **[Necessity]** Obvious from the result of Sontag and Wang.
[Sufficiency] The following Sontag-type controller makes the system ISS.

Sontag-type controller for ISS stabilization

$$u = \alpha(x) = \begin{cases} -\dfrac{\omega + \sqrt{\omega^2 + \{(L_gV)(L_gV)^\top\}^2}}{(L_gV)(L_gV)^\top}(L_gV)^\top & (L_gV \neq 0) \\ 0 & (L_gV = 0), \end{cases}$$

$$\omega = L_fV + \|L_{g_w}V\|\rho^{-1}(\|x\|).$$

Necessary and sufficient condition for ISS stabilization (2)

We will obtain $\dot{V}$ under the Sontag-type controller.

- **Cases for $L_g V \neq 0$:** For (x, w) such that $\|x\| \geq \rho(\|w\|)$,

$$\dot{V} \leq \|L_{g_w} V\| \{\|w\| - \rho^{-1}(\|x\|)\} - \sqrt{\omega^2 + \{(L_g V)(L_g V)^\top\}^2} < 0$$

- **Cases for $L_g V = 0$:** For (x, w) such that $\|x\| \geq \rho(\|w\|)$ and $x \neq 0$,

$$\dot{V} \leq L_f V + \|L_{g_w} V\| \rho^{-1}(\|x\|) < 0 \quad \Longleftarrow \text{Condition of ISS-clf}$$

Hence, the closed-loop system is ISS.
Moreover, from the scp condition we can show that the control law is continuous, as well as the cases of usual stabilization.