Frontiers of System Creation Technologies (システム創成学特論)

Fundamentals of Robust Control

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Robust Control

The part provided by Systems Control Theory Laboratory of lectures of ‘Frontiers of System Creation Technologies’ (システム創成学特論) explain an introduction of robust control.

Schedule

- **The first half** (Prof. Yamashita; 山下 裕): Robustness, robust stability, H_∞-norm, L_2-gain, Riccati-equation
- **The latter half** (Prof. Kobayashi; 小林 孝一): Linear matrix inequality (LMI), Schur compliment, Conversion of Riccati-equation, Various constraints and their LMI expressions

What is ‘Robustness ?’

Robustness in dictionary

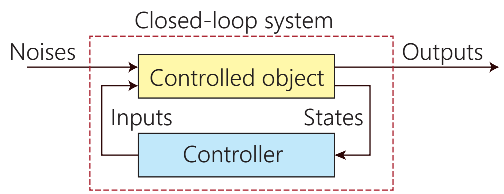
- The quality or condition of being strong and in good condition.
- The ability to withstand or overcome adverse conditions or rigorous testing.

For control systems,

- Qualities to be protected are
  Stability, Output Precesion, Fast convergence, ...
- Adverse conditions are
  System perturbation, Disturbance (e.g. Noises), ...

Evaluations of gains

To evaluate the effects of the disturbances on the output, some concepts of gains are useful, which are indices of the robustness on the accuracy of an output against disturbances.



Assumption

The closed-loop system is stable.

Gain for the angular frequency ω: $\|G(j\omega)\|$

We consider the following two cases:

- **White noise case:** A white noise includes all frequency components evenly. ⇒ The power spectrum of the white noise is flat.
- **Worst disturbance case:** A disturbance that includes only the following frequency component is worst for the output:

$$\omega_{\text{worst}} = \underset{\omega}{\operatorname{argmax}} \|G(j\omega)\|.$$

White noise

Fourier transform cannot be applied to white noises

- White noise: $w(t)$ ($\mathbf{E}[w] = 0$, $\mathbf{E}[w(t)w(t+\tau)] = \sigma^2\delta(\tau)$) $\Rightarrow$
  $$w_T(t) = \begin{cases} w(t) & (0 \le t \le T) \\ 0 & (\text{otherwise}) \end{cases}$$
- Fourier transform: $W_T(\omega) = \int_0^T w(t)e^{-j\omega t}dt$ (It exists!)
- Inverse Fourier transform: $w_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W_T(\omega)e^{j\omega t}d\omega$
- Power spectrum density (PSD): $S(\omega) = \lim_{T\to\infty}\frac{1}{T}W_T(\omega)^* W_T(\omega) = \sigma^2$ (Theorem of Wiener-Khintchine)
- Average L_2 norm of $w(t)$ does not exist:
  $$\|w(\cdot)\|_{2,\text{ave}}^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T |w(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega = \infty$$

H_2-norm

Consider an SISO case with a white noise.
System: $G(s)$, Input: $w(t)$, Output: $z(t)$
We assume that the initial state is at the origin.

- Average L_2 norm of the output $z(t)$: ($z_T(t)$: output under $w = w_T(t)$)
  $$\|z(\cdot)\|_{2,\text{ave}}^2 = \lim_{T\to\infty}\frac{1}{T}\int_0^T |z(t)|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{\infty} |z_T(t)|^2 dt$$
- Parseval's theorem:
  $$\|z(\cdot)\|_{2,\text{ave}}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 \cdot \underbrace{\lim_{T\to\infty}\frac{1}{T}|W_T(\omega)|^2}_{=\sigma^2(\text{const})} d\omega$$
  $\Rightarrow$ PSD of output
- Therefore, $\frac{\|z(\cdot)\|_{2,\text{ave}}^2}{\sigma^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2 d\omega$

Definition of H_2-norm of stable SISO systems

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2 d\omega}$$

H_2-norm for MIMO case

We can extend the notion of H_2-norm to MIMO systems.

Definition of H_2-norm of stable MIMO systems

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\text{trace}[G(j\omega)^* G(j\omega)]d\omega}$$

It means the ratio between σ, which indicates the amplitude of the noise, and the average L_2-norm of $z(t)$

$$\|z(\cdot)\|_2 = \sqrt{\lim_{T\to\infty}\frac{1}{T}\int_0^T z(t)^\top z(t)dt}$$

when $w(t)$ is a vector of white noises.

Worst disturbance

Consider more general disturbance.

When the L_2-norm of the disturbance is fixed as

$$\|w(\cdot)\|_2^2 = \int_0^{\infty}|w(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|W(j\omega)|^2 d\omega = \sigma^2,$$

we consider the maximization of

$$\|z(\cdot)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2 \cdot |W(j\omega)|^2 d\omega.$$

Worst disturbance for the output

$$|W(j\omega)|^2 = 2\pi\sigma^2\frac{\delta(\omega - \omega_{\text{worst}}) + \delta(\omega + \omega_{\text{worst}})}{2}$$
$$\omega_{\text{worst}} = \underset{\omega\ge 0}{\text{argmax}}|G(j\omega)|$$

H_∞-norm

Under the worst disturbance, $\|z(\cdot)\|_2/\|w(\cdot)\|_2$ coincides with the maximum value of $|G(j\omega)|$.

Assumptions

- $G(s)$ is stable.
- Initial condition: $x(0) = 0$.

H_∞-norm for SISO systems

$$\|G(s)\|_\infty = \sup_\omega |G(j\omega)| \quad \left(= \sup_{\|w\|_2 \neq 0, w \in L_2} \frac{\|z(\cdot)\|_2}{\|w(\cdot)\|_2} \right)$$

H_∞-norm for MIMO systems

$$\|G(s)\|_\infty = \sup_\omega \max_i \sigma_i[G(j\omega)] \quad \left(= \sup_{\|w\|_2 \neq 0, w \in L_2} \frac{\|z(\cdot)\|_2}{\|w(\cdot)\|_2} \right)$$

$\sigma_i[G]$: i-th singular value of a matrix G

Another meaning of H_2-norm

For SISO systems, H_2-norm satisfies the following equation:

$$\|G(s)\|_2 = \sup_{\|w\|_2 \neq 0} \frac{\|z(\cdot)\|_\infty}{\|w(\cdot)\|_2} = \sup_{\|w\|_2 \neq 0} \frac{\sup_T |z(T)|}{\|w(\cdot)\|_2}$$

Proof: Let $g(t)$ denote the impulse response of $G(s)$. Then,

$$|z(T)|^2 = \left| \int_0^T g(t) w(T-t) dt \right|^2 \le \int_0^T g(t)^2 dt \cdot \int_0^T w(t)^2 dt$$
$$\le \|G(s)\|_2^2 \cdot \|w(\cdot)\|_2^2$$

holds. For the disturbance $w(t) = g(T-t)/\sqrt{\int_0^T g(\tau)^2 d\tau}$, the above inequality becomes an equality $|z(T)| = \sqrt{\int_0^T g(t)^2 dt}$. Note that $\|w(\cdot)\|_2 = 1$. By making $T \to \infty$, we obtain $|z(T)| \to \|G(s)\|_2$.

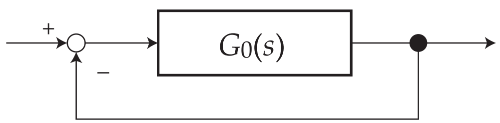
Robust stability

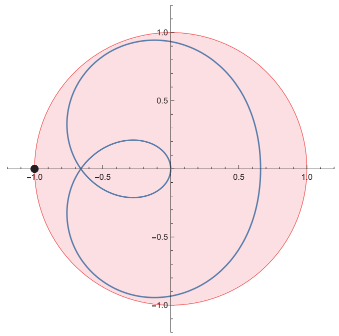
Robust stability

- The property that the stability is robust against system perturbation.
- Several stability margins are proposed.
- Theoretically, the robust stability can be explained by the notion of H_∞-norm and small-gain theorem.

Small gain theorem

Let $G_0(s)$ be a stable transfer function.





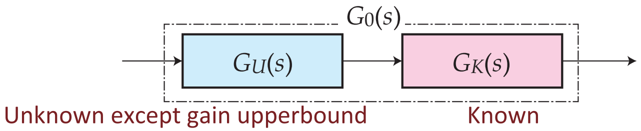
Small gain theorem

The above closed-loop system is stable, if $\|G_0(s)\|_\infty < 1$.

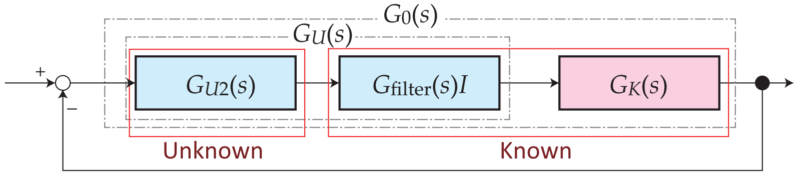
For SISO systems, the small gain theorem can be proven by the notion of Nyquist plot.

S-G theorem for cascaded system with uncertainty

We assume that $G_0(s)$ is decomposed as $G_0(s) = G_U(s)G_K(s)$, where $G_U(s)$ is an unknown stable transfer-function (square) matrix except an upperbound of its gain $L(\omega) \geq \sigma_{\max}[G_U(j\omega)]$.



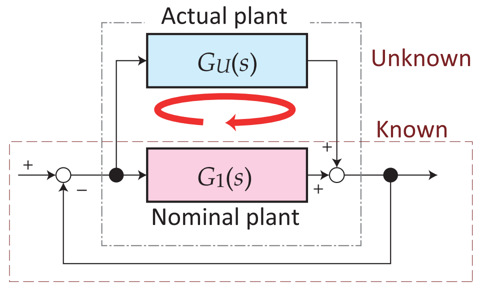
Suppose that there exists a stable transfer function $G_{\text{filter}}(s)$ such that $|G_{\text{filter}}(j\omega)| = L(\omega)$. Then, we obtain $G_0(s) = G_{U2}(s) \cdot \big(G_{\text{filter}}(s)G_K(s)\big)$, where $\|G_{U2}(s)\|_\infty \leq 1$.

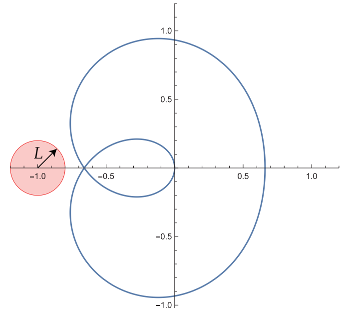


Stability condition of the closed-loop system: $\|G_{\text{filter}}(s)G_K(s)\|_\infty < 1$

System including uncertainty

Consider the case where the actual plant consists of a nominal plant $G_1(s)$ and stable unknown part $G_U(s)$. The upper bound of $\|G_U(s)\|_\infty$ is known as $L(\omega)$.





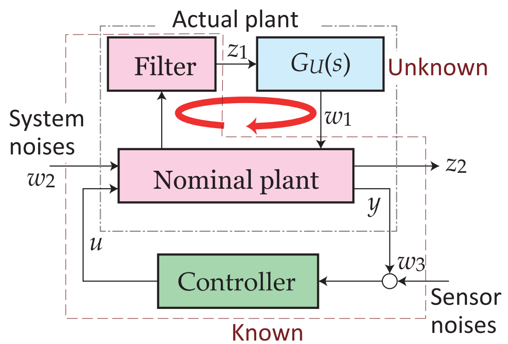
SISO-system case where $L(\omega)$ is constant

Small gain theorem derives the following condition.

Stability condition

- Nominal closed-loop system $G(s)(I + G(s))^{-1}$ is stable.
- $L(\omega)\sigma_{\max}[(I + G_1(j\omega))^{-1}] < 1 \Leftrightarrow L(\omega) < \sigma_{\min}[I + G_1(j\omega)]$ $({}^\forall\omega \in \mathbb{R})$

More general case



- In state-feedback cases, $y = x$ and $w_3 = 0$.
- It is assumed that the nominal closed-loop system is stable.
- $\|G_U(s)\|_\infty \leq 1$

- **Robust stability condition**: $[H_\infty$-norm from w_1 to $z_1] < 1$
- **Performance criterion**: (γ: small value) $[H_2$ or H_∞-norm from (w_2, w_3) to $(z_2, ku)] \leq \gamma$
- $\Rightarrow$ **Combined condition**: $[H_\infty$-norm from (w_1, w_2, w_3) to $(z_1, \gamma^{-1}z_2, k'u)] < 1$

Notations

- We denote a transfer matrix $G(s) = C(sI - A)^{-1}B + D$ as

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \quad \left(\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \end{aligned}\right)$$

- A rational-function matrix $G(s)$ is called **proper**, if $\sigma_{\max}[G(\infty)] < \infty$.

$G(s)$ is proper $\Leftrightarrow$ $G(s)$ can be expressed as $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

- $G(s)$ belongs to RH_∞, if $G(s)$ is a stable proper rational-function matrix.

$$G(s) \in RH_\infty \Leftrightarrow G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \ \mathrm{Re}\,\lambda[A] < 0$$

Lyapunov equation

Positive definite matrix

A real symmetric matrix P is called to be positive definite, if $x^\top P x > 0$ for all x $(\neq 0)$.

- A matrix P is positive definite, if and only if all eigenvalues of P are positive.
- The positive definiteness of P is simply denoted by $P > 0$.

Theorem

A linear autonomous system $\dot{x} = Ax$ is (globally) asymptotically stable, if and only if, for a positive matrix Q, there exists a positive definite matrix P such that

$$PA + A^\top P = -Q \quad \text{(Lyapunov equation)}.$$

Lyapunov function: $V(x) = x^\top P x > 0, \quad {}^\forall x \neq 0$

$$\dot{V} = x^\top (PA + A^\top P)x = -x^\top Qx \leq -\min_i \lambda_i(Q)\|x\|^2 \leq -\frac{\min_i \lambda_i(Q)}{\max_i \lambda_i(P)} V$$

$\Longrightarrow V(x(t)) \to 0 \ (t \to \infty) \Longrightarrow x(t) \to 0 \ (t \to \infty).$

Conditions of H_∞-norm in state space (1)

We investigate a condition of $\|G(s)\|_\infty \leq \gamma$ for a fixed γ, in the state-space expression.

$$\|G(s)\|_\infty \leq \gamma \Leftrightarrow \frac{\|z(\cdot)\|_2}{\|w(\cdot)\|_2} \leq \gamma \Leftrightarrow J = \int_0^\infty z^\top z - \gamma^2 w^\top w dt \leq 0 \quad (x(0) = 0)$$

Worst disturbance w: A disturbance w that maximizes J.

Assumption: $D = 0$

Riccati equation and worst disturbance

Riccati equation: $A^\top X + XA + \gamma^{-2} XBB^\top X + C^\top C = 0, \quad X > 0$

Worst disturbance: $w^* = \dfrac{1}{\gamma^2} B^\top X x$

Conditions of H_∞-norm in state space (2)

If a positive-definite solution X of the Riccati equation exists, then

$$x(T)^\top X x(T) - x(0)^\top X x(0) = \int_0^T (Ax + Bw)^\top X x + x^\top X(Ax + Bw) dt$$

$$= \int_0^T w^\top B^\top X x + x^\top X B w - x^\top C^\top C x - \gamma^{-2} x^\top X B B^\top X x dt$$

$$= \int_0^T -\gamma^2 (w - w^*)^\top (w - w^*) - x^\top C^\top C x + \gamma^2 w^\top w dt$$

$$\leq \int_0^T -\|z\|^2 + \gamma^2 \|w\|^2 dt = -J.$$

Therefore, L_2-gain condition is satisfied when $x(0) = 0$.

- The uniqueness of the solution of the Riccati equation is not guaranteed.
- Internal stability under the disturbance $w = w^*(x)$ is not guaranteed.
- A solution X under which the system with $w = w^*$ is stable is called a **stabilizing solution**.
- A stabilizing solution is positive definite.

Conditions of H_∞-norm in state space (3)

- The Riccati equation can be relaxed as a Riccati inequality.
- The result can be extended to the cases with $D \neq 0$.

Theorem

The following three conditions are equivalent:

1. $\|C(sI - A)^{-1}B + D\|_\infty < \gamma$
2. $\gamma^2 I - D^\top D > 0$, and there exists a positive-definite solution X of Riccati inequality

$$A^\top X + XA + (XB + C^\top D)(\gamma^2 I - D^\top D)^{-1}(B^\top X + D^\top C^\top) + C^\top C < 0.$$

3. The following LMI (Linear Matrix Ineqality) holds:

$$\begin{bmatrix} XA + A^\top X & XB & C^\top \\ B^\top X & -\gamma I & D^\top \\ C^\top & D & -\gamma I \end{bmatrix} < 0, \quad X > 0.$$

Conditions of H_2-norm in state space

H_2-norm can be also obtained by state-space calculation.

- The stability of the system is assumed.
- $D = 0$ is assumed. (H_2-norm does not exist, when $D \neq 0$.)

Observability Gramian:

$$L_O = \int_0^\infty e^{A^\top t} C^\top C e^{At} dt > 0$$

It can be obtained from a Lyapunov equation

$$A^\top L_O + L_O A + C^\top C = 0$$

H_2-norm calculation in the state space

$$\|G(s)\|_2 = \sqrt{\int_0^\infty \mathrm{trace}[B^\top e^{A^\top t} C^\top C e^{At} B] dt} = \sqrt{\mathrm{trace}[B^\top L_O B]}$$

H_∞ control problem (1)

Controlled object:

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = Cx + D_1 w + D_2 u$$

- $x \in \mathbb{R}^n$: state, $w \in \mathbb{R}^m$: disturbance (noise)
  $u \in \mathbb{R}^\ell$: control input, $z \in \mathbb{R}^p$: evaluation output

Problem setting (in frequency domain)

Obtain a state feedback $u = \alpha(x)$ that makes H_∞-norm from w to z less than or equal to a given positive value γ.

Assumptions: To simplify the problem, we make the following assumptions.

$D_1 = 0$, $C^\top D_2 = 0$ (Condition of orthogonality), $\mathrm{rank}\, D_2 = \ell$
(A, B_2): Stabilizable, (A, C): Detectable

H_∞ control problem (2)

Under the assumptions, there exists an orthogonal matrix T ($T^\top T = I$) such that

$$z = Cx + D_2 u = T\left(\underbrace{\begin{bmatrix} C_0 \\ 0 \end{bmatrix} x}_{\text{Term for control performance}} + \underbrace{\begin{bmatrix} 0 \\ D_{20} \end{bmatrix} u}_{\text{Term for input magnitude}} \right).$$

The L_2-norm of z becomes

$$\|z(\cdot)\|_2 = \sqrt{\int_0^\infty (Cx + D_2 u)^\top (Cx + D_2 u) dt}$$
$$= \sqrt{\int_0^\infty x^\top C^\top C x + u^\top D_2^\top D_2 u dt}$$

H_∞ control problem (3)

The problem is equivalent to the following new problem:

Problem setting (in time domain)

Obtain a feedback $u = K_2 x$ that makes the performance criterion

$$J(x_0, w, u) = \int_0^\infty \|z(\tau)\|^2 - \gamma^2 \|w(\tau)\|^2 \, d\tau$$

non-positive for all $w(\cdot)$, when $x(0) = 0$.

From assumptions,

$$J(x_0, w, u) = \int_0^\infty x^\top C^\top C x + u^\top D_2^\top D_2 u - \gamma^2 w^\top w \, dt$$

Zero-sum differential game

Zero-sum differential game on H_∞ control

- One player who can manipulates u aims to minimize J.
- Another player who can manipulates w aims to maximize J.

Problem

Find **optimal strategy (control law) for players**

$$w = K_1^* x \quad (\text{worst disturbance})$$
$$u = K_2^* x \quad (\text{optimal input})$$

such that

$$J(x_0, w, K_2^* x) \le J(x_0, K_1^* x, K_2^* x) \le J(x_0, K_1^* x, u), \quad {}^\forall w, {}^\forall u \in \mathfrak{U}(x_0, K_1^*),$$

if they exist.

$\mathfrak{U}(x_0, K_1^*)$: Set of $u(\cdot)$ such that $x \to 0$ $(t \to \infty)$ under $w = K_1^* x$.

Solution of the differential game

Riccati equation

$$XA + A^\top X + C^\top C + X\left(\frac{1}{\gamma^2} B_1 B_1^\top - B_2 R^{-1} B_2^\top\right) X = 0, \quad X \succ 0$$

- $R = D_2^\top D_2$ $(\succ 0)$
- Multiple positive definite solution may exist.
- We adopt a **stabilizing solution** X such that $A + (1/\gamma^2) B_1 B_1^\top X - B_2 R^{-1} B_2^\top X$ is stable.

Solution of the differential game

$$w = K_1^* x = \frac{1}{\gamma^2} B_1^\top X x$$
$$u = K_2^* x = -R^{-1} B_2^\top X x$$

Riccati inequality

When we only need

- L_2-gain from w to z that is less than or equal to γ, and
- Stability when $w = 0$,

the Riccati equation can be relaxed to a Riccati inequality

$$XA + A^\top X + C^\top C + X\left(\frac{1}{\gamma^2} B_1 B_1^\top - B_2 R^{-1} B_2^\top\right) X \preceq 0, \quad X \succ 0,$$

and its solution does not have to be a stabilizing solution.

- Under the feedback $u = -R^{-1} B_2^\top X x$,

$$x(T)^\top X x(T) - x(0)^\top X x(0) + \int_0^T \|z\|^2 - \gamma^2 \|w\|^2 dt \le 0, \quad {}^\forall w$$

  holds. $\Rightarrow$ L_2-gain condition is satisfied when $x(0) = 0$.
- For a Lyapunov function $V(x) = x^T X x$, when $w = 0$, $\dot{V} \le -\|z\|^2$ holds. Hence, from the detectability, the system is stable.

Summary

- H_2-norm evaluates the gain for the white noises.
- H_∞-norm evaluates the gain for a worst disturbance.
- H_∞-norm can be identified as an L_2-gain in state space.
- Robust stability condition can be converted to an H_∞-norm condition via the small-gain theorem.
- The H_∞-norm condition can be expressed by a solvability condition of a Riccati equation (inequality).
- H_∞ control problem can be solved via a Riccati equation (inequality) also.